

# Analytical Solution of a Spatial SIR epidemic model with memory inside a circular habitat with endemic memory using CAS.

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*Abstract – Using Computer Algebra Software (CAS) we obtained the reproduction rate of an Epidemic Propagation in a circular habitat, for two cases, in the first one the boundary has an endemic condition, in the other one it presents an infectious transference from other territories through its boundary. We transformed the initial equation into Laplace Domain, and then we solved the simplified differential equation found and then we calculated residues to achieve the solution. After a stability analysis we obtained the reproductive rate  $R_0$ . The method that was used can be extended to more complex problems such as indirectly transmitted diseases with one or more intermediary hosts or effects of genetic, immunological, geographical or social heterogeneity in the human population. This application indicates that the computer algebra software for symbolic computation has a very promissory future in mathematical epidemiology.*

**Keywords: Reproductive rate, CAS, Laplace transform, residue, epidemic threshold, memory.**

## 1. Introduction

Idealizing a habitat, which presents an epidemic, to a circular region, we formulate the propagation equation in its domain. We decided to solve this equation using a Computer Algebra Software (CAS). The explicit equation for the disease propagation is

$$\begin{aligned} & \left( \frac{\partial}{\partial t} f(r, t) \right) - \frac{\delta \left( \left( \frac{\partial}{\partial r} f(r, t) \right) + r \left( \frac{\partial^2}{\partial r^2} f(r, t) \right) \right)}{r} - (\beta_1 S_0 - \gamma_1) f(r, t) \\ & - \beta_2 S_0 \int_0^t f(r, \tau) e^{(-\varepsilon_1(t-\tau))} d\tau + \gamma_2 \int_0^t f(r, \tau) e^{(-\varepsilon_2(t-\tau))} d\tau \\ & - \delta_1 \int_0^t \frac{\left( \left( \frac{\partial}{\partial r} f(r, \tau) \right) + r \left( \frac{\partial^2}{\partial r^2} f(r, \tau) \right) \right) e^{(-\varepsilon_0(t-\tau))}}{r} d\tau = 0 \end{aligned} \quad (1)$$

Where  $f(r,t)$  represents the density of infectious individual at the point of polar coordinate  $(r,\theta)$  at the time  $t$ , with  $0 \leq r \leq a$ .

$\beta_1$  is the infectiousness,  $\gamma_1$  is the removal rate of infected individuals,  $\gamma_2$  is the memory of recovery,  $\beta_2$  is the memory of infectiousness,  $\delta_1$  is the memory of diffusion and  $\varepsilon_k$  the decay factor of memories (with  $k=0..2$ ).

We consider here the following two cases.

**CASE 1**

with the initial condition

$$f(r, 0) = 0 \tag{2}$$

and with the boundary Dirichlet's condition

$$f(a, t) = \mu_b e^{(-\eta t)} \tag{3}$$

**CASE 2**

with the initial condition

$$f(r, 0) = 0 \tag{4}$$

and with the boundary Neumann's condition

$$\lim_{r \rightarrow a} \frac{\partial}{\partial r} f(r, t) = \mu_b e^{(-\eta t)} \tag{5}$$

In case 1,  $\mu_b$  is the density of infectious individuals on the boundary and  $\eta$  is the decay rate of  $\mu_b$ .

In case 2,  $\mu_b$  is the infectious individual flow that cross the boundary and  $\eta$  is the decay rate of  $\mu_b$ .

The question here is analytically solve the equation (1) using symbolic computation assisted by CAS, so that we may obtain the basic reproductive rate of the disease in a circular habitat with the specified boundary condition. The present work is a prolongation and generalization of the results that were originally presented in [8].

**2. Methods**

The method that is applied here was proposed originally in [8]. We use Laplace transform technique and apply residue theory to and the inverse Laplace transform [2]. Unfortunately the actual CAS do not incorporate directly the inverse Laplace transform using of residue theory and therefore is necessary introduce such residue manually. The method of computation can be formulated at the following way:

- Apply the Laplace transform with CAS to equation (1) and reduce to an ordinary differential equation.
- Solve such ordinary equation with the boundary conditions using CAS
- Implement in CAS the necessary calculus of residues to obtain the inverse of the Laplace transform to obtain the analytical solution of (1) and verify the result.
- Extract from the solution, the explicit form of the reproductive rate of disease in a circular.

**3. Results**

**CASE 1**

The solution of the equation (1) with initial condition (2) and boundary condition (3) obtained using computer algebra software is

$$f(r, t) = \frac{\mu_b J_0(\lambda(-\eta) r)}{e^{(\eta t)} J_0(\lambda(-\eta) a)} + \left( \sum_{j=1}^4 \left( \sum_{n=1}^{\infty} \left( - \frac{e^{(S_{j,n} t)} \mu_b J_0\left(\frac{\alpha_n r}{a}\right)}{(S_{j,n} + \eta) J_1(\alpha_n) \left(\frac{d}{dS_{j,n}} \lambda(S_{j,n})\right) a} \right) \right) \right) \tag{6}$$

Where  $J_m$  is the first kind Bessel of order  $m$  and  $\alpha_n$  are zeroes of  $J_0$  [3] with  $1 \leq n < \infty$  y  $S_{j,n}$  are the solutions of the fourth degree equation  $\lambda(s) = \alpha_n/a$ .

We obtain from (6) the basic reproductive rate of the disease, denoted  $R_0$ :

$$R_{0,n} = \frac{S_0 a^2 \varepsilon_2 \varepsilon_0 (\beta_1 \varepsilon_1 + \beta_2)}{\varepsilon_1 (a^2 \gamma_1 \varepsilon_2 \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta_1 + a^2 \gamma_2 \varepsilon_0)} \quad (7)$$

The basic reproductive rate of the equation (7) is a generalization of the formula that gives the  $R_0$  corresponding to the temporal models without spatial considerations [1], [4], [7]. Here  $R_{0,n}$  is named epidemic reproductive rate, and the deterministic endemic threshold is  $R_{0,n} > 1$ .

## CASE 2

The solution of the equation (1) with initial condition (4) and the boundary condition (5) obtained using computer algebra software is

$$f(r, t) = -\frac{\mu_b J_0(\lambda(-\eta) r)}{e^{(\eta t)} J_1(\lambda(-\eta) a) \lambda(-\eta)} + \left( \sum_{j=1}^4 \left( -\frac{e^{(S_j, t)}}{(S_j + \eta) \lambda(S_j) \left( \frac{d}{dS_j} \lambda(S_j) \right) a} \right) \right) + \left( \sum_{j=1}^4 \left( \sum_{n=1}^{\infty} \left( -\frac{\mu_b J_0\left(\frac{\alpha_n r}{a}\right) e^{(S_{j,n}, t)}}{\lambda(S_{j,n}) (S_{j,n} + \eta) J_0(\alpha_n) \left( \frac{d}{dS_{j,n}} \lambda(S_{j,n}) \right) a} \right) \right) \right) \quad (8)$$

Where  $J_m$  is the first kind Bessel function of order  $m$  and  $\alpha_n$  are now the non vanishing zeroes of  $J_1$ , with  $1 \leq n < \infty$ .  $S_{j,n}$  y  $S_j$  are respectively the solutions of the fourth degree equations  $\lambda(s) = \alpha_n/a$  and  $\lambda(s) = 0$ .

We extract from (8) two basic reproductive rate denoted  $R_{0,n,a}$ ,  $R_{0,n,b}$ .

$$R_{0,n,a} = \frac{S_0 a^2 \varepsilon_2 \varepsilon_0 (\beta_2 + \beta_1 \varepsilon_1)}{\varepsilon_1 (a^2 \gamma_1 \varepsilon_2 \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta_1 + a^2 \gamma_2 \varepsilon_0)} \quad R_{0,n,b} = \frac{S_0 \varepsilon_2 (\beta_2 + \beta_1 \varepsilon_1)}{\varepsilon_1 (\gamma_2 + \gamma_1 \varepsilon_2)} \quad (9)$$

The basic reproductive rate of the equation (9) denoted  $R_{0,n,b}$  is the basic reproductive rate for purely temporal models without spatial effects but with memory. Here  $R_{0,n,a}$  is named epidemic reproductive rate, and the deterministic endemic threshold is  $R_{0,n,a} > 1$ .

We take  $R_{0,n,b}$  as the dominant instead of  $R_{0,n,a}$  because  $R_{0,n,b}/R_{0,n,a} \geq 1$  always. Then  $R_{0,n,b}$  is the dominant basic reproductive rate for the Neumann case.

For both cases we have that:

$$\lambda(s) = \sqrt{\frac{B(s)}{A(s)}} \quad A(s) = -\delta - \frac{\delta_1}{s + \varepsilon_0} \quad B(s) = s - \beta_1 S_0 + \gamma_1 - \frac{\beta_2 S_0}{s + \varepsilon_1} + \frac{\gamma_2}{s + \varepsilon_2}$$

## 4. Discussion and conclusions

We may see that the Dirichlet's condition case has a basic reproductive rate of the disease which is minor than the corresponding to the Neumann's conditions case.

From the other side is worthwhile to compare the results that were obtained here for the case of a circular habitat, with the results that correspond to the case of a linear habitat. Explicitly the profile of infected individuals for a one-dimensional habitat is the following when we consider only endemic boundaries at  $x=0$  and  $x=a$ :

$$\begin{aligned} f(x, t) = & \frac{-\mu_b \sin(\lambda(-\eta) x) \cos(\lambda(-\eta) a) + \mu_b \cos(\lambda(-\eta) x) \sin(\lambda(-\eta) a) + \mu_b \sin(\lambda(-\eta) x)}{e^{(\eta t)} \sin(\lambda(-\eta) a)} \\ & + \left( \sum_{i=1}^4 \left( \sum_{n=1}^{\infty} \frac{\mu_b e^{(t S_{i,n})} \sin\left(\frac{n \pi x}{a}\right) (-1 + (-1)^{(-n)})}{(S_{i,n} + \eta) \lambda_1(S_{i,n}) a} \right) \right) \end{aligned} \quad (10)$$

and the corresponding  $R_0$  is

$$R_{0,n,linear} = \frac{S_0 a^2 \varepsilon_2 \varepsilon_0 (\beta_2 + \beta_1 \varepsilon_1)}{\varepsilon_1 (n^2 \pi^2 \varepsilon_2 \delta_1 + n^2 \pi^2 \varepsilon_2 \delta \varepsilon_0 + a^2 \gamma_1 \varepsilon_2 \varepsilon_0 + a^2 \gamma_2 \varepsilon_0)}$$

As we saw, the basic reproductive rate in a circular habitat with Dirichlet condition is

$$R_{0,n} = \frac{S_0 a^2 \varepsilon_2 \varepsilon_0 (\beta_1 \varepsilon_1 + \beta_2)}{\varepsilon_1 (a^2 \gamma_1 \varepsilon_2 \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta \varepsilon_0 + \alpha_n^2 \varepsilon_2 \delta_1 + a^2 \gamma_2 \varepsilon_0)}$$

We can observe then that the linear model has a reproductive rate minor that the corresponding to the circular habitat because  $n^2 \pi^2$  is always more greater than  $\alpha_n^2$  For example some zeros of  $J_0$  are [3]

2.404825558, 5.520078110, 8.653727913, 11.79153444

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