Realization of Digital Fuzzy Operations Using Multi-Valued Fredkin Gates

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Abstract

Multi-valued Fredkin gates (MVFG) are reversible gates and they can be considered as modified version of the better known reversible gate the Fredkin gate. Reversible logic gates are circuits that have the same number of inputs and outputs and have one-to-one and onto mapping between vectors of inputs and outputs; thus the vector of input states can be always reconstructed from the vectors of output states. It has been shown that for power not to be dissipated in an arbitrary circuit it is necessary that the circuit be build from reversible gates. Moreover multi-valued Fredkin gates have been shown to be a suitable choice as a basic building block for binary and different alternative logics for example multi-valued logic and threshold logic. In this paper we show the application of MVFGs in the implementation of fuzzy set and logic operations. Fuzzy relations and their composition is very important in this theory as collections of fuzzy if-then rules and, fuzzy GMP (Generalized Modus Ponens)and GMT (Generalized Modus Tollens) respectively is mathematically equivalent to them. In this paper we describe digitized fuzzy sets where the membership values are discretized and represented using ternary variables and the implementation of set operations. The composition of fuzzy relations and a systolic array structure to compute it is described. Design with reversible gates and the highly parallel architecture of systolic arrays makes the proposed circuits quite attractive for implementation.

Key words: *Digital fuzzy sets, Fuzzy relation, Systolic array, Reversible gates, Multi-Valued Fredkin gate.*

1 Introduction

Fuzzy set theory, introduced by Zadeh [1], and the corresponding logic is quite transition from the traditional set theory and the concept of uncertainty. When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily true or false, it may be true only to some degree. It is most common to express the degrees of membership by numbers in the closed interval [0,1]. In this paper we consider digital fuzzy set where the membership-value space is discretized. We define the standard set operations and the concept of fuzzy relations based on these digital fuzzy sets and their realizations. In this paper we describe the composition of fuzzy relations and describe a systolic array structure to compute it. Collections of fuzzy *if-then* rules or fuzzy algorithms are mathematically equivalent to *fuzzy relations* and the problem of inference of (evaluating them with specific values) is mathematically equivalent to *composition* [2,3,4].

The proposed circuit is composed of Multi-Valued Fredkin Gates (MVFG) which are reversible gates. Conservative and reversible logic gates are widely known to be compatible with the new computing paradigms like optical and quantum computing. Reversible logic gates are circuits that have the same number of inputs and outputs and have one-to-one and onto mapping between vectors of inputs and outputs; thus the vector of input states can be always reconstructed from the vectors of output states. Irreversible functions (gates in the classical binary logic except the NOT gate are irreversible) can be converted into reversible functions easily. If the maximum number of identical output vectors is p, then $\lceil \log p \rceil$ garbage outputs (and some inputs, if necessary) must be added to make the input-output vector mapping unique. Reversible logic applicable to quantum computing [5], nanotechnology [6] and low power design [7]. In [7] Bennett showed that for power not to be dissipated in an arbitrary circuit it is necessary that the circuit be build from reversible gates. Multi-valued reversible gates however have not got much attention until recent times. In [8,9,10,11,12] we find some proposed some gates and some synthesis techniques. In this paper we concentrate on the multiple-valued Fredkin gates proposed by Picton in [11,12].

The organization of the paper is as follows: Section 2 describes the reversible logic, different gates, and the MVFG. Section 3 we provide the basic terminologies of the fuzzy set theory and fuzzy relations. We also describe the concept of digital fuzzy sets. In section 4 we describe the implementation of the basic fuzzy operations and the systolic array to compute the composition of fuzzy relations and section 5 concludes the paper.

2 Reversible Logic

The circuits proposed in this paper are composed with reversible gates. In the following subsections introduces some of the reversible gates and the MVFG which we are going to use extensively. Landauer [13] proved that traditional irreversible gates lead to power dissipation in a circuit regardless of its implementation. This theorem along with Bennett's theorem in [6] actually points out that to keep the Moore's law functioning the future technology should be based on reversible logic. A gate is reversible only when there is a one-to-one and onto relationship between the gates inputs and output which mean that these gates must have equal number of inputs and outputs. Only then the inputs of a reversible gate can be uniquely determined from the outputs. Thus a reversible gate with n inputs must have n outputs and this is denoted as a (n,n) or n*n logic gate. It can be pointed out that all classical gates e.g. AND, OR, XOR are irreversible. The NOT gate however can be considered as a (1,1) reversible gate. Irreversible functions can be converted into reversible functions easily. If the maximum number of identical output vectors is p, then $\lceil \log p \rceil$ garbage outputs (and some inputs, if necessary) must be added to make the input-output vector mapping unique. If we consider the EXOR operation we can easily see that the operation is an irreversible one. Now in Feynman gates there exists two output x′= x and y′= $x \oplus y$ for the two inputs x and y. The truth table given below shows that there exists the required unique input-output vector mapping.

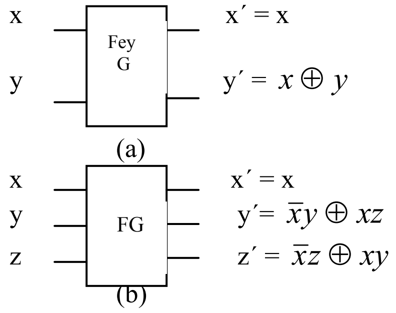
Table 1: Feynman Gate

X	Y	x′	y′
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Traditional logic design method differs significantly from the synthesis of reversible functions. Techniques for reversible synthesis can be found in [14,15,16] and the references within. Efficient design of adders using reversible gates has also gained much attention as found in [17,18,19].

2.1 Some Basic Reversible Gates and Classical Digital logic Using these Gates

Among many gates Fedkin gates together with Toffoli gates [20] and Fynman gates[21] are the most often discussed gates in reversible and quantum architecture and it is suggested that future realization efforts will concentrate mostly on these gates and their derivations. These reversible gates along with a new gate are shown below



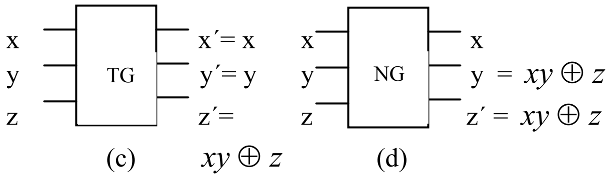


Figure 1: (a) Feynman gate, (b) Fredkin gate, (c) Toffoli gate and (d) New gate[19]

In strict reversible logic paradigm signal fan-out is forbidden. However most of the gates provide one of the inputs at the outputs unaltered. Using constant inputs we also can generate the fan-out and other different function. The following figure shows some such constructions

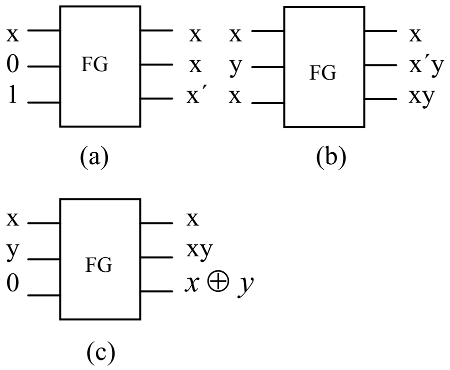
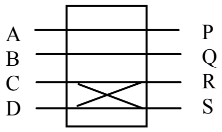


Figure 2: Basic Logic Operations Using Reversible Gates

In figure 2 (a) and (b) we use Fredkin gates to implement the fanout and AND operation. In (c) we find the operation of AND and EXOR on two inputs performed. It can be pointed out that for this gate the output z should be $x' \oplus y'$ which is equivalent to $x \oplus y$.

2.2 Multi-valued Fredkin Gate

Multi-valued Fredkin Gate (MVFG) was proposed by Picton in [11, 12]. In [11] he suggested as it is possible to implement any Boolean logic function using Fredkin gates then it is also possible using MVFGs as they are modified Fredkin gates. Figure 3 shows the MVFG describes by the following equations:



Where,
P = A
Q = B
R = C *if* A<B *else* R = D
S = D *if* A<B *else* S = C

Figure 3: Multiple-Valued Fredkin Gate (MVFG)

One observation can be made here is the fact that the definition of the gate does not specify the type of signals. Thus they can be binary, multi-valued etc. The only requirement is that the relation (<)

can be defined on them. It was suggested and shown by Picton that these gates can be used to implement alternative logic as threshold logic, array logic etc., and these gates can be constructed using optical devices such as photonic switches that are being developed in telecommunications.

3 Fuzzy sets and Relation

Now that we have introduced the different reversible gates in this section we discuss the fuzzy set operations and the fuzzy relation along with their composition operation.

Fuzzy sets: Zadeh [1] introduced fuzzy sets by defining characteristic functions for fuzzy sets that we call membership function as $\mu_A(x) : X \rightarrow [0,1]$. So in fuzzy sets we talk about the degree of that an element x can have denoted by $\mu_A(x)$ which is a number between 0 and 1. Membership functions thus may represent an individual's (subjective) notion of a vague class – for example tall people, little improvement, big benefit etc.

If X is a universe of discourse and x is a particular element of X then a fuzzy set A defined on X may be written as a collection of ordered pairs A= {(x, $\mu_A(x)$)}, x ε X. Alternatively a fuzzy set may be written as

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

if the universe of discourse is discrete and $\mu_A(x)$ in this case can be called a discrete-universal membership function and if we have a continuous universe of discourse we write

$$A = \int_X \mu_A(x) / x$$

The membership function defined above can be called a continuous-universal space membership function and fuzzy sets with such membership functions may be called *analog fuzzy sets*. Numerical processing using digital components requite finite data with finite precision. For such purposes we define digital fuzzy sets.

Digital fuzzy set: If a discrete-universal membership function can take only a finite number, n≥2 of distinct values then we call this fuzzy set a digital fuzzy set.

Thus for digital implementation, an analog fuzzy sets membership function is discretized along both the universal space and membership-value dimensions. Assume that the universal space is quantized into 16 discrete values and the membership-values can take n=8 distinct values then we need 16 x 3 = 48 bit to represent the set. In this paper we consider 9 distinct values represented by 2 ternary variables.

Example 1: Digital fuzzy set:
Suppose the membership function of a fuzzy set representing the concept of a middle-aged person is given as

$$M(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or} \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

Now a possible discrete approximation A(x):{0,5,10,15,….,80}→ [0,1] of the membership function can be defined in Table 2.

Table 2: Discrete Approximation

X	A(x)
x ε {25,30,…,55}	0.0
x ε {25,55}	0.33
X ε {30,50}	0.67
x ε {35,40,45}	1.00

Table 3: Digitized membership values

Encoding		Value
A_2	A_1	
0	0	0.0
0	1	0.15
0	2	0.3
1	0	0.4
1	1	0.5
1	2	0.6
2	0	0.7
2	1	0.85
2	2	1.0

Now suppose we define only 9 different levels of membership-values using 2 ternary variables as shown in Table 3 then we may represent the digital fuzzy set as D = 0.3/25 + 0.7/30 + 1/35 + 1/40 + 1/45 + 0.7/50 + 0.3/55. (End of example)

From this example it is possible to see that considering more elements and having a larger number of membership values we can more precisely represent a fuzzy set.

Fuzzy operations: There are 3 standard fuzzy set operations namely complement, intersection and union. We also discuss the concept of fuzzy relation and the composition operation.

Complement: Let A be a fuzzy set on X, then by the complement of A has the membership function A(x) = 1- A~~(x)~~, this value may be interpreted not only as the degree to which x belongs to A the complement of A but also as the degree to which x does not belong to A.

Intersection/ t-norm and Union/t-conorm: The intersection or the union of two fuzzy sets A and B is specified in general by a binary operation on the unit interval: i.e., a function of the form *f: [0,1] x [0,1] → [0,1].* For each element x of the universe set, this function produces the intersection as (A∩B)(x) = i[A(x),B(x)] = A(x) ^ B(x) and the union is expressed as (AUB)(x) = u[A(x),B(x)] = A(x) v B(x).

There exists different t-norm and t-conorm operators available, however the standard operation for intersection and union are the following

Standard intersection: i(a,b) = min(a,b), Standard union: u(a,b) = max(a,b) where a,b ε [0,1].

We will use the standard operations throughout the paper. For digital fuzzy sets it is easy to compute the complement operation- we just need to complement the bits. In section 4 we show the circuit construction of the complement or negate and, the min and max operations .

Fuzzy Relations: Fuzzy relations are fuzzy sets defined on Cartesian products. A binary fuzzy relation R defined on a discrete Cartesian product X×Y can be written as

$R = \Sigma\ \mu_R(x_i,y_i)/\ (x_i,y_i)$, where

every pair (x_i,y_i) ε X×Y.

We will be using digital fuzzy relations that is the $\mu_R(x_i,y_i)$'s can take only a fixed number of values. We can easily represent a fuzzy relation in a matrix form. A fuzzy relation on two sets X = $\{x_1,x_2,x_3,x_4\}$ and Y = $\{y_1,y_2,y_3,y_4\}$can be represented in a 4×4 matrix R where $R_{i,j} = \mu_R(x_i,y_j)$.

Composition of fuzzy relation: Given two fuzzy relations- R_1 on X×Y and R_2 on Y×Z we may define a new relation denoted as $R_1 \circ R_2$ on X×Z. There are several types of composition-namely max-min, max-product, max-average. The max-min composition formula is given below:

$$R_1 \circ R_2 \equiv \sum_{X\times Z} \bigvee_{y} [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]/(x,z)$$

.

We can see the computation of the membership grades is very much similar to matrix multiplication, with max (v) being analogous to summation and min (^) being analogous to multiplication.

Example 2: Max-Min Composition of Fuzzy Relations

Let R_1:

	z_1	z_2	z_3	z_4
y_1	0.9	0.0	0.3	0.4
y_2	0.2	1.0	0.8	0.0
y_3	0.8	0.0	0.7	1.0
y_4	0.4	0.2	0.3	0.0
y_5	0.0	1.0	0.0	0.8

R_2:

	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.2	0.0	1.0	0.7
x_2	0.3	0.5	0.0	0.2	1.0
x_3	0.8	0.0	1.0	0.4	0.3

Then according to the max-min composition $R_1 \circ R_2$:

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.7
x_2	0.3	1.0	0.5	0.8
x_3	0.8	0.3	0.7	1.0

(End of Example)

We intend to use this max-min composition because by far it is the most common type in engineering applications.

Compositions are very important for inferencing procedures used in linguistic description of systems and is particularly useful in fuzzy controllers and expert systems[2]. Collections of fuzzy *if-then* rules or fuzzy algorithms are mathematically equivalent to *fuzzy relations* and the problem of inference of (evaluating them with specific values) is mathematically equivalent to *composition.* Different fuzzy implication operator and relation between the fuzzy GMP(Generalized Modus ponens) and GMT(Generalized Modus tollens) and fuzzy relations and composition operation are described in detail in [3,4].

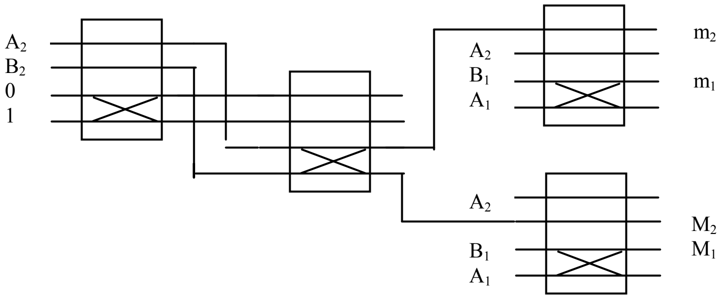


Figure 4: Implementation of Min and Max Operation Using MVFGs

In Section 4 we show a systolic array structure that can be used to compute composition of fuzzy relations. The cells, composed of reversible logic gates, are actually responsible for the max-min operations.

4 Proposed Circuits

The proposed circuits are based on the MVFGs described above. In section 5.1 we describe the circuit that computes the min and min operation on digitized fuzzy membership values. The negation operation is also implemented. Then in the next subsection we describe the systolic array structure built with basic min-max cell to compute the composition of fuzzy relations.

4.1 Fuzzy Operations using MVFG

In fuzzy set theory and fuzzy logic the min and max operations the most important and the most frequently used one. In this paper we are considering the membership-values to be digitized and represented by 2 ternary variables, thus there are 9 distinct membership levels. In [22] the min and max operations using MVFGs is shown, the problem with those circuit is that the way the membership values are taken makes the circuit for negation using MVFGs quite inefficient. That is why we chose ternary variables to represent the membership values. Use of multi-valued variables of with radix is possible but they make the circuits using more components.

Suppose we want to calculate min(A,B) or max(A,B), where $A = A_2A_1$ and $B = B_2B_1$ where A and B are represented by 2 ternary variables. The variable with subscript 2 is the most significant bit.

In figure 4 we first find the $\min(A_2, B_2)$ and $\max(A_2, B_2)$ and then use them to produce $m_2m_1 = \min(A,B)$ and $M_2M_1 = \max(A,B)$.

Next in figure 5 we show the implementation of the complement operation. We can actually perform this operation digit-wise.

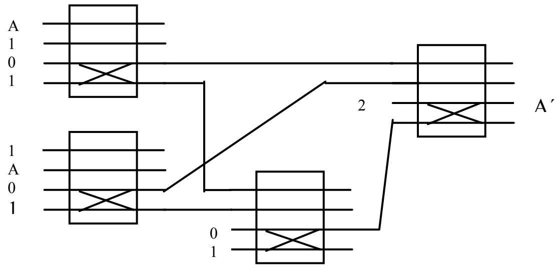


Figure 5: Complementing a Ternary Variable using MVFGs

For example if a membership values is represented by A_2A_1 where $A_2 = 2$ and $A_1 = 1$ [representing say 0.75, see example 1], the membership in the complement fuzzy set should be $A_2`A_1`$ where $A_2` = 0$ and $A_1` = 1$ [representing 0.25]. In the following figure we show the complementation of a single ternary variable using MVFGs.

4.2 Systolic Array Structure for Composition of Fuzzy Relations

Systolic arrays are data-processing circuit formed by interconnecting a set of identical data-processing cells in a uniform manner. Data word flow synchronously from cell to cell, where a small step of the overall function is performed, until the results emerge from the boundary cells. It provides a high degree of parallelism and the use of identical cells and uniform interconnections making then ideal for implementation. In figure 6 we show the basic cell structures with its inputs and outputs. These cells are connected in a uniform manner to compute the fuzzy relation operation as shown in figure 5. The cell simply computes $z = z' \vee (x \wedge y)$, where z' is the value computed from the adjacent cell and x and y are the input membership values. The cells also propagate inputs and the value computed to the adjacent cells as shown in figure 7.

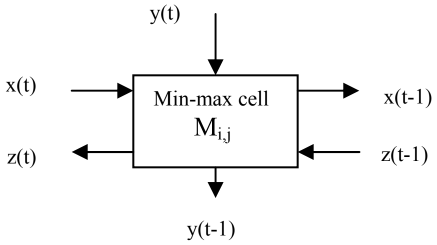


Figure 6: Basic Cell

In figure 7 we have shown the array that can be used to compute the composition of two relations represented by n×n matrices. It is important to realize that we have to take care so that data is input in correct sequence.

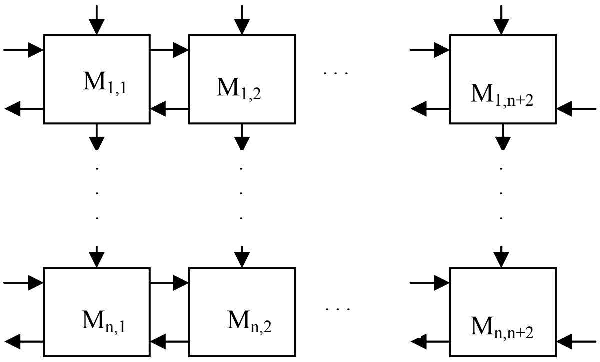


Figure 7: Systolic Array for Composition of Fuzzy Relation

5 Conclusion

We in this paper introduce the digitized fuzzy sets and discuss the different operations. Compositions of fuzzy relations are described. Compositions are very important for inferencing procedures used in linguistic description of systems and are particularly useful in fuzzy controllers and expert systems. Collections of fuzzy *if-then* rules or fuzzy algorithms are mathematically equivalent to *fuzzy relations* and the problem of inference (evaluating them with specific values), fuzzy GMP (Generalized Modus Ponens) and GMT (Generalized Modus Tollens) is mathematically equivalent to composition. We show a systolic array structure for the computation of composition of fuzzy relations. It provides a high degree of parallelism and the use of identical cells and uniform interconnections making then ideal for implementation.

In this paper we continue with the new logic design paradigm – reversible logic. The reversible logic finds its application in many fields such as quantum and optical computing, low power design, nanotechnology etc. The proposed design utilizes the multi-valued reversible logic gates [namely the multi-valued fredkin gate(MVFG)]. Not many circuit design techniques have appeared in the literature concerning multi-valued reversible gates or the implementation of fuzzy operations using them. Future research on this topic is necessary to compare different multivalued reversible logic gates as the basic building blocks. However Fedkin gates together with Toffoli gates and Fynman gates are the most often discussed gates in reversible and quantum architecture and it is suggested that future realization efforts will concentrate mostly on these gates and their derivations. As it is possible to implement any Boolean logic function using Fredkin gates then it is also possible using MVFGs as they are modified Fredkin gates. This along with the fact that multiple-valued Fredkin gates can be used to implement alternative logics(for example threshold logic) makes MVFGs a rather attractive choice.

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