Statistical Analysis of Linear Random Differential Equation

Seifedine Kadry
IFMA, France – LaMI Laboratory
skadry@gmail.com

Abstract - *In this paper, a new method is proposed in order to evaluate the stochastic solution of linear random differential equation. The method is based on the combination of the probabilistic transformation method for a single random variable and the numerical methods (e.g. finite difference, finite element, Runge-Kutta, etc…). The transformation technique evaluates the probability density function (PDF) of the solution by multiplying the PDF of the random variable by the Jacobean of the inverse function*

Keywords: *Transformation method, numerical methods, random variable, random differential equation.*

1 Introduction

A random differential equation (RDE) is a differential equation containing a random term. The study of RDE is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of differential equations.

Random differential equations appear in several different applications: study of random evolution of systems with a spatial extension(random interface growth, random evolution of surfaces, fluids subject to random forcing), study of stochastic models where the state variable is infinite dimensional(for example, a curve or surface).

The solution to random differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). In the case where the RDE is an evolution equation, the finite dimensional points of view consists in viewing the solution at a given time as a random element in a space function and thus view the RDE as a stochastic evolution equation in an infinite dimensional space. In the path wise point of view, tries to give a meaning to the solution for (almost) every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined.

In this work, a new technique is proposed in order to evaluate the probability density function of the solution, based on the combination of the probabilistic transformation methods and the numerical methods.

2 Stochastic Methods

The solution of a stochastic differential equation is gotten when we evaluate the probability density function of this solution. We can use several methods, for example Fokker-Planck equation [3], development of Wiener-Hermit [6], method of perturbation [1], method of stochastic local linearization [5], method of decomposition [7], stochastic finite element method [2] and other.

2.1 Equation of Fokker-Plank

The answer of a mechanical system in a dynamic structure, with parameters and vague excitations, are determined in general by using the methods of the vague vibrations. Most of these excitations form an uncertain process, and more precisely a white noise or Gaussian. This choice of the noise transforms our system to a Markov process, which we can express according to the density of transition probability that is adapted by the equation of the Fokker-Planck.

Example: Consider the following system

$$mx'' + cx' + kx = F \qquad (1)$$

In this case, EFP:

$$\frac{\partial p}{\partial t} = y\frac{\partial p}{\partial x} + \frac{kx}{m}\frac{\partial p}{\partial x} + \frac{cy}{m}\frac{\partial p}{\partial y} + \frac{\partial^2 p}{\partial x^2} \qquad (2)$$

With y = x'.

We can solve (2) analytically Wang and Uhlenbeck, numerically by finite element method [3].

2.2 Development of Wiener-Hermit

The development of Wiener-Hermit transforms a stochastic equation in a system determined by partial differential equations [6].
Example: Let u(x) is any vague function; the development of Wiener-Hermit of u(x) is:

$$u(x)=\int K^{(1)}(x-x_1)H^{(1)}(x_1)dx_1+\int\int K^{(1)}(x-x_1)H^{(1)}(x_1)dx_1 \qquad (3)$$

With H polynomial of Hermit [6] defined by
$H^1(x) = a(x)$, $H^2(x_1,x_2) = a(x_1,x_2)$,……

2.3 Stochastic linearization method

The theory of this method expresses the stochastic development of Ito-Taylor according to the multiple integrals of Wiener [5].

2.4 A stochastic finite element method

The stochastic finite element method consists of representing in a probabilistic form the solution of a linear random differential equation. Each input random variable is expanded into a Hermite polynomial series in standard normal random variables. The response or the solution is expanded onto the so-called polynomial chaos. The coefficients of the expansion are obtained by a Galerkin-type method in the space of probability [2].

3 Transformation Method

The transformation technique evaluates the PDF (Probability Density Function) of the system output(solution of RDE) by multiplying the PDF of the input(random variable) by the Jacobean of inverse transformation [4][9][10], which can be determined either analytically or numerically. This approach has the advantage of giving directly the whole density function of the solution, which is very helpful for statistical analysis. Let *y' = f(x)*, a random differential equation has one unknown, therefore

$$PDF(y)=\left|J\right|PDF(x) \qquad (4)$$

With

$$J=\begin{vmatrix} \frac{\partial y'_1}{\partial x_1} & . & . & . & \frac{\partial y'_1}{\partial x_n} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ \frac{\partial y'_1}{\partial x_1} & & & & \frac{\partial y'_1}{\partial x_n} \end{vmatrix}$$

We have used this technique in this article with the numerical methods to evaluate the *PDF* of the solution of *RDE*.

4 Algorithm of the proposed method

Step1- Solve the differential equation *y'=f(x)* (in term of the random variable)Using one of the numerical methods.
Step2- Find the inverse function $x=f^{-1}(y)$
Step3- Find the Jacobean and its determinant
Step4- Calculate the *PDF* of the solution using

$$PDF(y)=\left|J\right|PDF(x)$$

5 Applications

I) let us solve the following random differential equation:

$$y'=kx \quad with\ y_0=1\ and\ k\ uniform\ random\ variable(k\Rightarrow U(1,2))$$

$$\Rightarrow y=\frac{kx^2}{2}+c$$

$$\Rightarrow 1=c$$

$$\Rightarrow y=\frac{kx^2}{2}+1 \Leftarrow step1$$

$$\Rightarrow k=\frac{2(y-1)}{x^2} \Leftarrow step2$$

$$\Rightarrow \left|J\right|=\frac{2}{x^2} \Leftarrow step3$$

$$\Rightarrow PDF(y)=\frac{2}{x^2}PDF(k) \Leftarrow step4$$

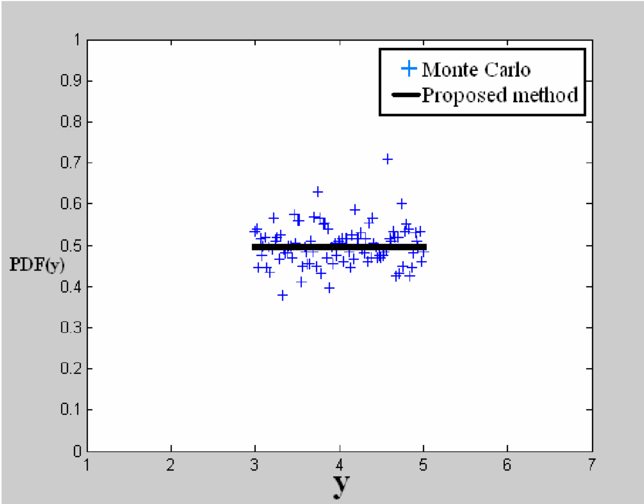

Figure 1. Probability Density Function of y when k è U(1,2)

II) Non homogeneous RDE:

Let us solving the following non homogeneous random differential equation, using our new technique:

$$y' = y + x^2 \quad with\ y_0 = k,\ \ k\ normal\ random\ variable$$

$$\Rightarrow y' - y = x^2$$

$$\Rightarrow \times e^{-x} \Rightarrow e^{-x}(y' - y) = x^2 e^{-x}$$

$$\Rightarrow (ye^{-x})' = x^2 e^{-x}$$

$$\Rightarrow y = Ce^{x} - (x^2 + 2x + 2)$$

$$\Rightarrow C = k + 2$$

$$\Rightarrow y = (k+2)e^{x} - (x^2 + 2x + 2)$$

$$\Rightarrow k = e^{-x} y + e^{-x}(x^2 + 2x + 2) - 2$$

$$|J| = -e^{-x}$$

$$\Rightarrow PDF(y) = e^{-x} PDF(k)$$

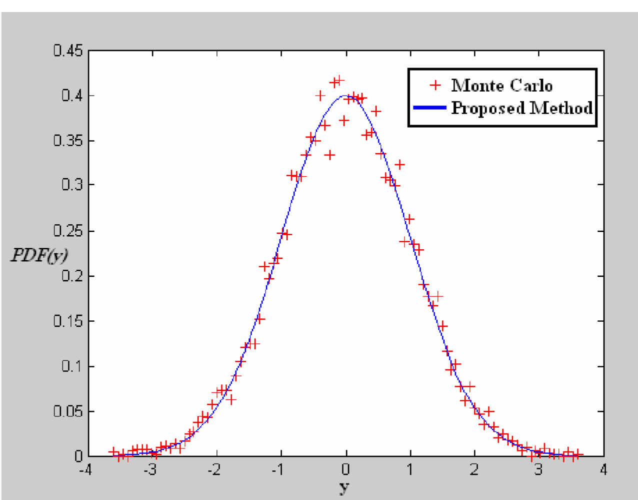

Figure 2. Probability Density Function of y when k è N(0,1)

III) Runge-Kutta method

Using Runge-Kutta method to solve the following equation:

$$\frac{dy}{dx} = y + kx^3,\ y(0) = 1\ and\ k\ exponential\ random\ variable (k \Rightarrow \exp(1))$$

If we apply the Runge-Kutta method with *h=1*, then

$$y(1) = y_1$$

$$\Rightarrow$$

$$m_1 = f(0,1) = 1$$

$$m_2 = f(\frac{1}{2}, \frac{3}{2}) = \frac{1}{8}k + \frac{3}{2}$$

$$m_3 = f(\frac{1}{2}, \frac{15}{8}) = \frac{1}{8}k + \frac{15}{8}$$

$$m_4 = f(1, \frac{25}{8}) = k + \frac{25}{8}$$

$$\Rightarrow y_1 = 1 + \frac{1}{6}(1 + \frac{1}{4}k + 3 + \frac{1}{4}k + \frac{15}{4} + k + \frac{25}{8})$$

$$\Rightarrow y_1 = 1 + \frac{1}{48}(24k + 87) = \frac{1}{2}k + \frac{135}{48}$$

$$\Rightarrow k = 2y_1 - \frac{135}{24} \Rightarrow |J| = 2 \Rightarrow PDF(y_1) = 2PDF(k)$$

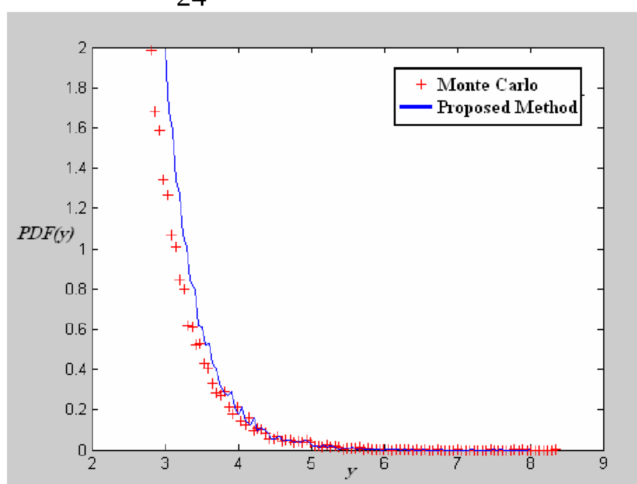

Figure 3. Probability Density Function of y when k è exp(1)

IV) Finite-Element method

Now, we want to use the Finite Element Method to solve the following RDE:

$$\begin{cases} -y'' + cy = f(x) \quad if\ \ 0 < x < 1 (c \Rightarrow U(0,1)) \\ y(0) = y(1) = 0 \end{cases}$$

If we divide the domain into four points, we obtain the following matrices [8],

$$A = \begin{bmatrix} 6 + \frac{c}{3} & -3 \\ -3 & 6 + \frac{c}{3} \end{bmatrix},\ \ Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\ \ and\ \ F = \frac{1}{3}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \Rightarrow AY = F \Rightarrow Y = A^{-1}F$$

For the second node we have $\Rightarrow y_2 = \frac{1}{3}\frac{(30+\frac{4c}{3})}{(6+\frac{c}{3})^2-9}$

$0.3358 < y_2 < 0.3448$ è

$$PDF(y_2) = \frac{-2}{(19y_2-4)y_2^2}[I_1]$$

$$I_1 = \log\left[\frac{9y_2+2}{19y_2-4}\right][171y_2-36]-687.78y_2+214.90$$

$0.3448 < y_2 < 0.3602$ è $PDF(y_2) = \frac{4.9732}{y_2^2}$

$0.3602 < y_2 < 0.3703$ è

$$PDF(y_2) = \frac{18}{(9y_2-2)y_2^2}[I_2]$$

$$I_2 = \log\left[\frac{9y_2+2}{9y_2-2}\right][9y_2-2]-39.48y_2+12.77$$

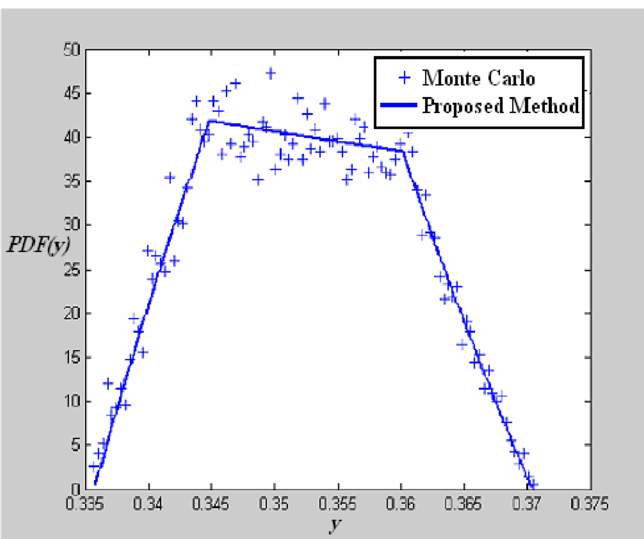

Figure 4. Probability Density Function of y_2 when c è *U(0,1)*

6 Conclusion

In this article, we are analyzing a differential equation with random variable. Our new technique based on the combination of the transformation method with numerical methods to evaluate the Probability Density Function (PDF) of the solution. Then to proof the performance of our method we compared the result with the result of 10000 simulation of Monte Carlo method.

7 References

[1]. G. Falsone and N. Impollonia, A new approach for the stochastic analysis of finite element modelled structures with uncertain parameters. Comput. Methods Appl. Mech. Engrg. 191 (2002) 5067–5085

[2]. B. Sudret & all, Eléments finis stochastiques en élasticité linéaire. C. R. Mecanique 332 (2004) 531–537

[3]. S.F. Wojtkiewicz and L.A. Bergman: Numerical solution for high dimensional Fokker-Planck equations. ASME 2001.

[4]. Numerical recipes en C: the art of scientific computing (Chapter 7: transformation method)

[5]. J.C. Jimenez et al.: simulation of stochastic differential equations through the local linearization method. Journal of statistical physics 1999.

[6]. T. Y. Hou and B. Wetton, Convergence of a finite difference scheme for the Navier-Stokes equations using vortices boundary condition, SIAM 1992.

[7]. S. Champier, Résolution du problème de Wicksell par une méthode de décomposition en Ondelettes Vaguelettes. UPRES-EA-3058 Avril 2000.

[8]. T. Gmür, Dynamique des structures. Presses Polytechniques et Universitaires Romandes. 1997.

[9]. H Hsu, Probability Random Variables and Random Processes. McGraw-Hill 1996.

[10]. M. Spiegel, Probability and Statistics. McGraw-Hill 2000.