Simulations Of Roll Waves Flows In Environmental Settings

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Abstract - In this paper we investigate the generation and structure of roll waves flows down a porous inclined plane and subjected to an applied surface shear stress. The unsteady equations of motion for the fluid layer are depth integrated according to the von Kármán momentum integral method accounting for the variation of the velocity distribution with depth. The slip boundary condition at the interface between the fluid layer and the porous plane is based on the assumption that the flow through the porous medium is governed by Darcy's law, and that the characteristic length scale of the pore space is much smaller than the depth of the fluid layer above. The instability of the uniform flow and the evolution to roll waves flow is determined by numerically solving the time dependent governing equations and calculating the nonlinear evolution of infinitesimal disturbances. Conclusions are drawn regarding the effect of the permeability of the porous inclined plane and its correlation with the effect of the shear stress applied on the surface of the fluid.

key-words: flow down a porous incline, nonlinear stability analysis, roll waves.

1 Introduction

Free-surface gravity-driven flows down an incline can exhibit a surface wave pattern which is referred to as "roll waves". This wave system results from the instability of the uniform flow and consists of progressive bores connected by sections of gradually varying flow.

Roll waves are observed frequently in flows found in the environment [1]. For example, sufficiently rapid mud flows develop a series of intermittent bores which can be capable of inflicting sever property damage and even topographical changes.

Due to the ubiquitous nature and relevant impact of roll waves flows, it is important to theoretically determine under what conditions roll waves will be generated and predict their structure and velocity of propagation. Such theoretical investigations can be conducted by first establishing a mathematical model for flow down an incline. This model can be used to simulate the evolution of a uniform and steady flow perturbed by a small disturbance. If the perturbed flow develops into a secondary quasi-steady flow the simulation depicts the expected roll waves flow.

In this paper we derive a model that incorporates certain factors not included in previous investigations which will allow us to obtain more realistic simulations of roll waves flows in environmental settings. In particular, we include the effect of wind stress by applying a prescribed shear stress at the surface of the fluid layer. This effect is coupled with that of the permeability of the inclined surface over which the fluid is flowing. This extension is important due to the fact that environmental flows mostly occur over porous surfaces such as soil, river beds, and layers of gravel [2].

2 The Mathematical Model

Consider the two-dimensional laminar flow of a thin layer of a Newtonian fluid along the surface of a porous medium incline at an angle θ with respect to the horizontal. We define an (x, z) coordinate system with the x-axis along the bottom and the z-axis normal to it. We denote the velocity and the total pressure by $\mathbf{u} = (u, w)^T$ and p respectively. We assume the characteristic longitudinal length of the flow, L to be much larger than the characteristic depth, H, with the depth varying slowly in time and in the longitudinal direction.

The equations of motion for the layer are obtained from the Navier-Stokes equations by neglecting terms of $O(\epsilon^2)$ with $\epsilon = H/L$ being the aspect ratio, and are expressed as [1]

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + g\rho\sin\theta + \mu\frac{\partial^2 u}{\partial z^2} \tag{2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\cos\theta, \tag{3}$$

where g is the acceleration due to gravity, and ρ and μ are the mass density and the viscosity of the fluid respectively. In obtaining these equations we also assumed moderately high Re flows with $Re = \rho UH/\mu \sim O(1/\epsilon)$, with U being a characteristic velocity of the flow.

The effect of wind stress can be incorporated into the model by prescribing the shear stress, τ_w at the surface of the layer. If we neglect terms of $O(\epsilon^2)$, including those arising from the effect of surface tension, then the continuity of force condition at the surface reduces to

$$\left.\begin{array}{l} p = 0 \\ \mu\dfrac{\partial u}{\partial z} = \tau_w \end{array}\right\} \quad \text{at} \quad z = h(x,t). \tag{4}$$

The kinematic condition at the surface is given by

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{at} \quad z = h(x,t). \tag{5}$$

We assume that the porous medium is saturated with fluid and that the flow through it is governed by Darcy's Law. We note that even if the free flow is a high Reynolds number flow, the application of Darcy's Law to model the flow through the porous medium is validated by the assumption that the geometry of the pore space results in a low Reynolds number flow. The condition at the fluid layer-porous medium interface in this case has been investigated by Beavers and Joseph [3]. They propose that a boundary-layer is formed at the top of the porous medium and that the tangential velocity rapidly changes across this region from the Darcian velocity to the velocity at the interface. Beavers and Joseph verify experimentally that the appropriate condition for the tangential velocity at the interface is

$$\frac{\partial u}{\partial z} = \frac{\chi}{\sqrt{\kappa}}(u - u_p) \quad \text{at} \quad z = 0, \tag{6}$$

where κ is the permeability of the porous medium, χ is a dimensionless parameter dependent on the structure of the porous medium, and $\boldsymbol{u}_p = (u_p, w_p)^T$ is the Darcian mean filter velocity in the porous medium. The normal velocity does not vary through the boundary-layer so the condition at the interface is

$$w = w_p \quad \text{at} \quad z = 0. \tag{7}$$

We will consider cases when the pore space geometry is such that the flow is slower than that of the layer above to the extent that the mean filter velocity terms in the interface conditions (6) and (7) are negligible [4]. The conditions at the bottom of the fluid layer then reduce to

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial z} = \dfrac{\chi}{\sqrt{\kappa}}u \\ w = 0 \end{array}\right\} \quad z = 0. \tag{8}$$

Integrating equation (3) and using the boundary condition (4) we find the expression for the total pressure to be:

$$p = g\cos\theta\rho(h - z).$$

Inserting this expression into the $x-$momentum equation (2) we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = g\sin\theta - g\cos\theta\frac{\partial h}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} \, . \quad (9)$$

Due to the fact that we are dealing with a small aspect ratio flow which varies slowly in the longitudinal direction, we can employ von Kármán's momentum integral method. Consequently, we assume that the dependence of the fluid velocity with depth is the same as that of a uniform and steady flow and integrate the governing equations over the depth of the fluid. Now, a uniform and steady flow of the fluid layer is given by

$$h = H = \text{const.}$$

$$u = u_0(z) = 3\bar{u}_0\frac{2(H+T)(z+\Gamma) - z^2}{2H^2 + 3TH + 6\Gamma(H+T)} \, , \quad (10)$$

where

$$\Gamma = \frac{\sqrt{\kappa}}{\chi} \quad \text{and} \quad T = \frac{\tau_w}{\rho g \sin\theta} \, ,$$

and $\bar{u}_0$ is the depth-averaged velocity given by

$$\bar{u}_0 = \frac{1}{H}\int_0^H u_0(z)dz.$$

Depth-integrating the continuity equation (1) and the momentum equation (9) over the fluid layer we obtain equations governing the depth of the fluid layer, $h(x,t)$ and the depth-averaged velocity, $\bar{u}(x,t)$. We can nondimensionalize these equations by introducing the nondimensional quantities (signified by an asterisk) according to

$$x = Lx^*, \quad \bar{u} = \bar{u}_0\bar{u}^*, \quad t = \frac{L}{\bar{u}_0}t^*, \quad h = Hh^*,$$

where $\bar{u}_0$ and H are the velocity and thickness of the uniform steady flow respectively, and

$$L = \frac{\bar{u}_0^2}{g\sin\theta}.$$

Employing this transformation to nondimensionalize the equations for h and $\bar{u}$ and dropping the asterisks for notational convenience we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} = 0 \quad (11)$$

$$\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left[\psi^2\phi + \frac{2}{3}\psi_0^2\Lambda h^2\right] = (h+\tau)\left(1 - \frac{\psi}{\psi_0}\right) \quad (12)$$

with

$$\psi = \frac{3\bar{u}}{2h^2 + 3\tau h + 6\gamma(h+\tau)},$$

$$\phi = \frac{8}{15}h^5 + \left(\frac{5}{3}\tau + \frac{8}{3}\gamma\right)h^4 + \left(4\gamma^2 + \frac{4}{3}\tau^2 + \frac{20}{3}\gamma\tau\right)h^3 + \left(4\gamma\tau^2 + 8\gamma^2\tau\right)h^2 + 4\gamma^2\tau^2 h \, ,$$

and where

$$\psi_0 = \frac{3}{2 + 3\tau + 6\gamma(1+\tau)} \, ,$$

$$\Lambda = \frac{\cot\theta}{Re} \, , \quad Re = \frac{g\rho^2 H^3 \sin\theta}{3\mu^2} \, ,$$

$$\gamma = \frac{\sqrt{\kappa}}{\chi H} \, , \quad \text{and} \quad \tau = \frac{\tau_w}{\rho g H \sin\theta} \, .$$

In the case when $\tau_w = 0$ and $\gamma = 0$ the Reynolds number can be expressed as $Re = \frac{\rho\bar{u}_0 H}{\mu}$.

The parameters in the governing equations (11) and (12) are Λ, γ, and τ. The parameter Λ is the ratio of the viscosity, which is a stabilizing factor, to destabilizing factors such as the thickness of the uniform fluid layer, the fluid mass density, and the inclination of the plane. The parameters γ and τ are the scaled permeability of the porous inclined plane and the scaled applied surface shear stress respectively.

3 Roll Waves Calculations

Our aim is to determine which uniform flows are unstable and evolve into roll waves flows

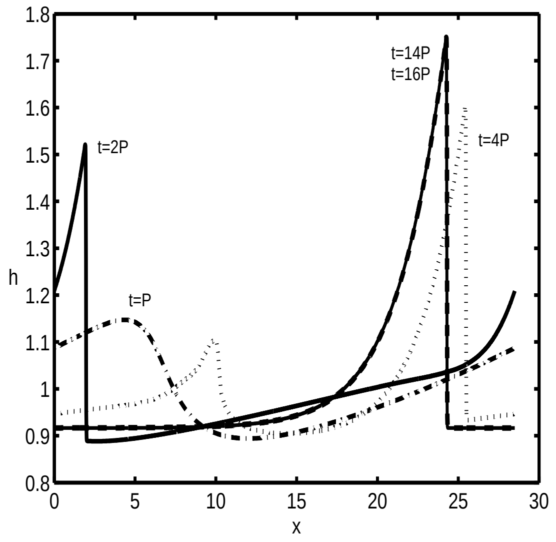

Figure 1: The evolution of the perturbed flow to roll waves solution with $\Lambda = 0.63$, $\tau = 0.3$, $\gamma = 0$, $P = 8.91$, and $\lambda = 28.5$.

and calculate various aspects of the wave structure such as the wave height, i.e. the height of the bores, the wavelength, and the speed of propagation. This is accomplished by calculating the evolution of the perturbed uniform flow over a periodic domain $x \in [0, \lambda]$. An example of such a calculation is illustrated by the results in figure 1. Here we present snapshots of the surface of the fluid at various times which are whole multiples of the period P. It can be seen that for times nP with n greater than approximately 14, the surface remains unchanged indicating that a roll waves solution is attained, with wavelength equal to the length of the calculation domain, λ and propagating with speed λ/P.

The governing equations (11) and (12) form a system of nonlinear hyperbolic conservation laws with a source term. To solve this system we employ the quasi-steady wave-propagation algorithm developed by Leveque [5]. This method is particularly effective in the context of stability analysis since it captures the balance between the source term and the flux gradient which is appropriate for cases when the solution is "quasi-steady". The discretization of the equations is accomplished by a second-order accurate finite-volume TVD scheme.

In terms of the nondimensional variables the change in mechanical energy across a bore is given by

$$\dot{E} = \left[\frac{1}{2} \int_0^h u^3 dz - \frac{1}{2} c \int_0^h u^2 dz + (\bar{u}h^2 - \frac{c}{2}h^2) \frac{4}{3} \Lambda \psi_0^2 \right]_2^1 ,$$

where $[f]_2^1 \equiv f_1 - f_2$ with f_1 denoting the limit of the quantity f as the bore is approached from the right and f_2 the limit of f as the bore is approached from the left. The speed of propagation of the bore is given by $c = [h\bar{u}]_2^1 / [h]_2^1$. The two integrals in the expression for $\dot{E}$ can be expressed in terms of $\bar{u}$ and h using the assumed velocity profile given by

$$u = 3\bar{u} \frac{2(h+\tau)(z+\gamma) - z^2}{2h^2 + 3\tau h + 6\gamma(h+\tau)} .$$

From the numerical solution to the governing equations we have the values of $\bar{u}$ and h, and we can thus calculate the energy drop across the bore in the roll waves solution. This quantity must be non-positive since mechanical energy is dissipated into heat due to turbulence. Since this mechanism is not accounted for in our model, the criterion of non-positive energy generation across the bore can be used to dismiss certain solutions as being non-physical.

Our numerical experiments reveal that for given values for the parameters γ and τ, there is a critical value of the uniform flow parameter $\Lambda = \Lambda_{\text{crit}}$, such that for the range $[0, \Lambda_{\text{crit}}]$ the uniform flow is unstable with the instability giving rise to roll waves flows, while for $\Lambda > \Lambda_{\text{crit}}$ roll waves do not form. Furthermore, for a given value of Λ in the unstable range various roll waves solutions exist with different wavelengths. It turns out that the energy drop across the bore decreases with wavelength, and there is a critical wavelength for which the energy drop is zero. This is the shortest physically possible roll waves solution. This solution is referred to as the "minimum roll wave" and corresponds to the wave pattern expected to emerge spontaneously from an unstable flow.

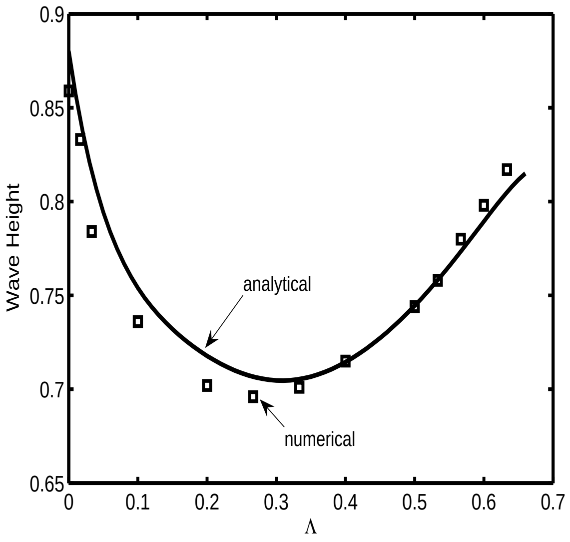

Figure 2: Wave height of the minimum roll wave as a function of Λ with $\tau = 0$ and $\gamma = 0$.

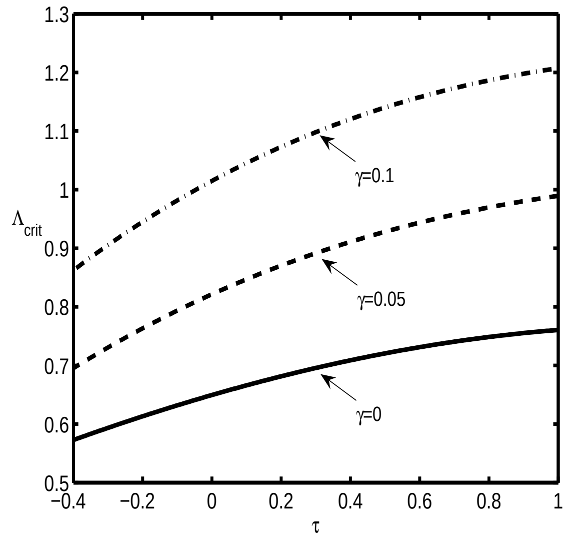

Figure 3: Critical value of Λ for formation of roll waves as a function of τ.

Ng and Mei [1] have previously studied roll waves generation on fluid layers with zero shear stress prescribed at the surface and flowing down an impermeable incline. They carry out an analytical investigation by seeking roll waves solutions characterized as periodic shocks connected by smoothly increasing depth profiles. They present an analytical procedure for calculating the wavelength, speed, and wave height of the roll waves. As is illustrated in figure 2, the results of this procedure are in good agreement with our numerical solutions for cases with $\gamma = 0$ and $\tau = 0$. In addition, the theoretically determined wave speed of the minimum roll wave in these cases agrees with experimental findings [6].

4 Results and Conclusions

In figure 3 we present the variation of Λ_{crit} with τ for different values of γ. It can be seen that the region of instability increases with τ and with γ. We can thus conclude that the applied shear stress destabilizes the flow when acting downhill and acts to stabilize the flow when applied uphill (negative values of τ). We can also deduce that the effect of the permeability of the inclined plane is to destabilize the flow. Physically, this behavior is to be expected since we know that the effect of friction is to retard the flow and thus act as a stabilizing factor. Increasing the permeability of the porous substrate diminishes the friction of the fluid with the bottom as, due to the slip condition, the velocity gradient at the bottom of the fluid layer decreases as the permeability, κ is increased.

In figures 4 - 6 we illustrate the effect of Λ and τ for a nonzero value of γ on the wave height, speed, and wavelength of the minimum roll wave respectively. The results presented in figure 4 indicate that there is significant variation of the wave height with τ for small values of Λ, which corresponds to highly unstable uniform flows. In particular, it can be seen that in these conditions the wave height decreases with τ. Now, as indicated by the results in figure 3, increasing the applied surface shear stress speeds up the flow and thus increases the interval of instability for the uniform flow parameter Λ. However, the results presented in figure 4 suggest that applying a surface shear stress in the downhill direction to highly unstable flows has a damping effect on the roll waves, with the wave height decreasing as a function of the magnitude of the surface stress. When

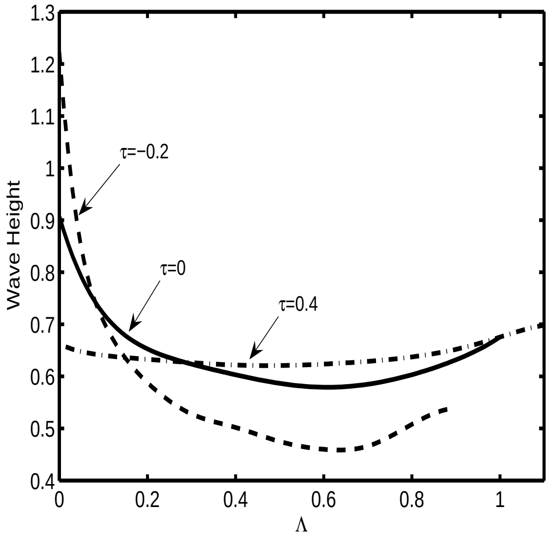

Figure 4: Wave height of the minimum roll wave as a function of Λ with $\gamma = 0.1$.

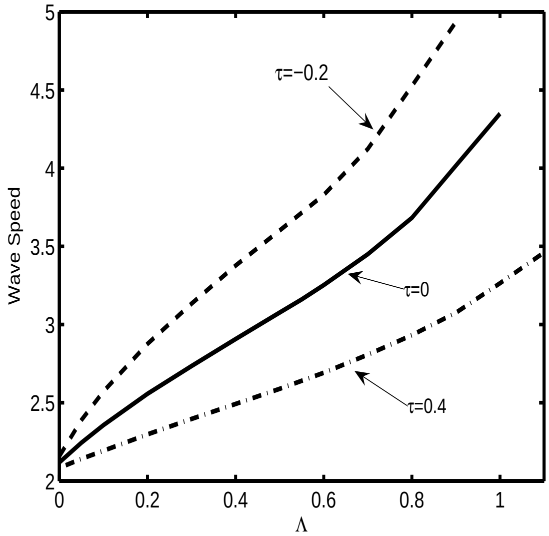

Figure 5: Wave speed of the minimum roll wave as a function of Λ with $\gamma = 0.1$.

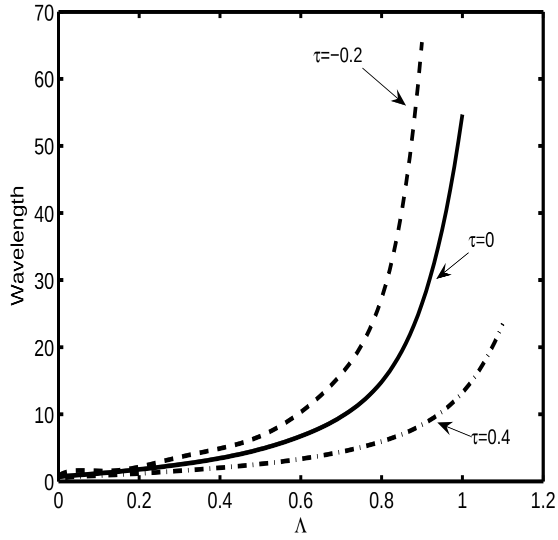

Figure 6: Wavelength of the minimum roll wave as a function of Λ with $\gamma = 0.1$.

the surface shear stress is applied in the uphill direction the opposite occurs, as the stress has a greater effect on slowing down the flow ahead of the bores. This results in a greater buildup of fluid at the bores as the magnitude of the stress is increased and thus a greater wave height.

The effect of the applied surface stress on the speed and wavelength of the minimum roll wave appears to be more pronounced for flow conditions near criticality (i.e. for the larger admissible values of Λ). By examining figure 5 we see that as Λ increases the variation of the wave speed with τ increases with the speed being a decreasing function of τ. A similar behavior is illustrated in figure 6 for the wavelength. In particular, we can conclude that under critical conditions for the onset of roll waves, the wavelength of the minimum roll wave decreases sharply as the applied surface shear stress is increased.

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5 References

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