Lyapunov Exponents and Information Dimension of Nonlinear Railway Wheelsets Incorporating Randomness

M.L. Liu
Department of Mechanical Engineering
Lakehead University
955 Oliver Road, Thunder Bay
Ontario, Canada P7B 5E1

J. Yu
Department of Mechanical Engineering
Lakehead University
955 Oliver Road, Thunder Bay
Ontario, Canada P7B 5E1

Abstract - *This article presents a numerical simulation that aims to examine the effect, of randomness in forward speed, in lateral clearance (dead band) and in both, on the dynamic behaviors of single- and two-axle railway wheelsets. Randomness is represented by pseudo-random numbers. They are incorporated into the dynamic models of the wheelsets. Subsequently, the temporal average of Lyapunov exponents is determined. The ensemble average of these exponents is then used to compute the information dimension. It is found that the introduction of small to moderate randomness does not necessarily lead to a chaotic response in the wheelset; in fact, the presence of small to moderate randomness may suppress chaotic response otherwise existing.*

Keywords: Nonlinear dynamic system, Lyapunov exponent, information dimension, randomness, railway wheelset.

1 Introduction

In the investigation of railway vehicle dynamics, wheel-rail interaction has drawn significant amount of attention. From the mathematics/modeling point of view, the equations of motion are highly nonlinear and the system is stiff; from the engineering point of view, wheelsets support the rail cars. Wheel-rail interaction, when properly understood, provides us with the basis of designing for safe, fast and pleasant movement of railway vehicle.

A good number of publications examined the nonlinear dynamics of wheelsets by identifying chaotic motion. Measures such as Poincare maps, Lyapunov exponents and bifurcation diagrams were commonly adopted. However, to the authors' best knowledge, most investigation treated the systems as deterministic, although in realty, the flange clearance (dead band) is varying along the tracks due to uneven wear, for example. The forward speed (traveling speed) is seldom constant. This calls for the need for introducing randomness to the wheelset dynamic model, giving rise to the present investigation.

The present investigation concerns itself with the dynamic behavior of railway wheelsets with randomness. Such dynamic behavior is mainly measured by Lyapunov exponents and information dimension. The wheelsets studied include both the single-axle and two-axle sets. The single-axle wheelset model employed is that used in [1] which incorporated the Vermeulen-Johnson contact theory [2] to account for the nonlinear creep force, whereas the two-axle model is that by Jensen *et al.* [3] which also incorporated the Vermeulen-Johnson contact theory.

The organization of the paper is as follows. Section 2 presents the fundamentals of Lyapunov exponents and information, followed by Section 3 which deals with random number generation. The mathematical models, results and discussions pertaining to the single- and two-axle wheelsets are then given in Sections 4 and 5. Conclusions are presented in Section 6.

2 Lyapunov exponents and information dimensions

For an autonomous dynamic system of degrees-of-freedom of n_D, introducing a state vector $Y = \left[X, \dot{X}\right]^T$ (where $X, \dot{X}$ are the displacement and velocity vectors of the dynamic system, respectively, and the superscript T indicates transpose), the equations of motion of the system are

$$\dot{Y} = f(Y, c) \tag{1}$$

with f being a vector of n functions and c a vector of parameters. Note that the dimension of Y is $n = 2n_D$. Denoting by $Y^*(t, c)$ the solution of Equ (1), and by $\Gamma(t)$ a vector representing the variation from Y^*, it can be shown that $\Gamma(t)$ satisfies [4]

$$\dot{\Gamma} = A(Y^*, c)\Gamma \tag{2}$$

where A is the Jacobian of f, that is $A_{ij} = \left.\frac{\partial f_i}{\partial Y_j}\right|_{Y=Y^*}$.

Measuring the i-th variation Γ_i ($i = 1, \ldots, n$) at time instants t_0, t_1, t_2, … , t_M , with M being a large integer, the i-th Lyapunov exponent λ_i can be determined by

$$\lambda_i = \frac{1}{t_M - t_0} \sum_{k=0}^{M-1} \log_2 \left| \frac{\Gamma_i(t_{k+1})}{\Gamma_i(t_k)} \right| \tag{3}$$

Since Γ is related to λ such that $\Gamma(t) = \Gamma(t_0) 2^{\lambda(t-t_0)}$ [4], a positive Lyapunov exponent indicates an exponentially increased Γ_i, and the existence of chaotic motion [4]. For a system whose state vector Y is of dimension *n*, the *n* Lyapunov exponents $\lambda_1, \lambda_2, ..., \lambda_n$ form the Lyapunov spectrum $\lambda_1 \geq \lambda_2 \geq \cdots\cdots \geq \lambda_n$ .Finally, the information dimension is [4]

$$d_I = J + \sum_{i=1}^{J} \lambda_i \Big/ |\lambda_{J+1}| \tag{4}$$

where the integer *J* is defined by the conditions $\sum_{i=1}^{J} \lambda_i > 0$ and $\sum_{i=1}^{J+1} \lambda_i < 0$. An information dimension that is not an integer, similar to a positive Lyapunov exponent, indicates the existence of chaotic motion [4].

3 Monte Carlo simulation

To introduce randomness to the dynamic system described by Equ (1), Monte Carlo simulation technique becomes necessary. Such technique usually involves generation of a series of pseudo- random numbers of certain distribution, which is the Gaussian distribution for the present study. Several random number generation techniques are implemented and tested. They fall broadly under two categories. Those in the first category rely on built-in pseudo-random number generators provided by compilers (such as Fortran or C) or software (such as Maple, MatLab, etc.) The quality of the generated pseudo-random numbers varies from good to poor. Those in the second category are the so-called portable random number generators, such as Netlib's Zufall. Two such generators are tested (Zufall and SM5 [5]). They prove to be of very good quality, see Table 1. The tests involve generating 25600 random numbers of standard normal distribution. The random numbers are then fed into the SPSS package so that mean, variance, skewness and kurtosis can be measured. The CPU times listed in Table 1 are those taken to generate 256 sets of 25600 random numbers on an IBM A31. SM5 requires longer time because each random number in the set is the mean of 24 random numbers of the same distribution. In what follows, only results associated with the Zufall random numbers are presented. In most cases, such results are very close to those associated with the SM5-generated random numbers.

Table 1 Comparison of pseudo-random number generators

Algorithm	SM5	Zufall
Mean	0.009235	-0.003700
Variance	1.0455	1.0040
Skewness	0.02	0.00
Kurtosis	-0.01	-0.02
CPU time (seconds)	22.7327	2.9242

4 Single-axle wheelsets

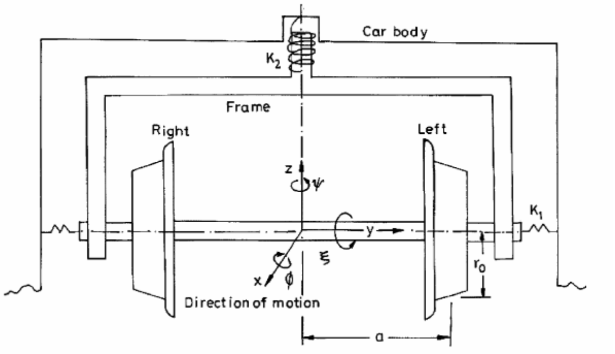

Figure 1 A single-axle wheelset

Table 2 Parametric values of the single-axle wheelset

m = 1022 kg	k_0 = 14.60 MN/m
I = 678 kg m^2	k_1 = 18.23 kN/m
a = 0.716 m	k_2 = 180 N/m
d_1 = 0.620 m	ψ= 0.54219
G = 808 MN/m^2	Φ = 0.60252
a_e = 0.006578 m	r_0 = 0.4572 m
b_e = 0.003934 m	λ = 0.05
δ = 0.0091 m	μN =10 kN

The wheelset has two wheels rigidly connected by one axle (see Figure 1 for geometry and Table 2 for parametric values). It is assumed to have two degrees-of-freedom, the lateral motion Y and yaw motion ψ. The two equations of motion are coupled through the nonlinear creep forces. The equations of motion are [1]

$$\begin{gathered} m\left(\frac{d^2Y}{dt^2}\right) + 2k_1Y + 2F_Y + F_T(Y) = 0\,, \\ I\left(\frac{d^2\psi}{dt^2}\right) + 2k_2d_1^2\psi + 2aF_X = 0 \end{gathered} \tag{5}$$

where the contact forces F_X and F_Y are

$$F_X = \frac{\xi_X}{\Phi}\frac{F_R}{\xi_R}\,,\ F_Y = \frac{\xi_Y}{\Psi}\frac{F_R}{\xi_R} \tag{6}$$

and the flange force is

$$F_T(Y) = \begin{cases} k_0(Y-\delta) & \delta < Y \\ 0 & -\delta \leq Y \leq \delta \\ k_0(Y+\delta) & Y < -\delta \end{cases} \tag{7}$$

In Equ (6), the resultant creep force is

$$F_R = (\mu N)\begin{cases} u - \frac{1}{3}u^2 + \frac{1}{27}u^3 & u < 3 \\ 1 & u \geq 3 \end{cases} \tag{8a}$$

with $u = \dfrac{G\pi a_e b_e}{\mu N}\xi_R$ and

$$\xi_X = \frac{\lambda Y}{r_0} + \frac{a\dot{\psi}}{v},$$

$$\xi_Y = \frac{\dot{Y}}{v} - \psi, \tag{8b}$$

$$\xi_R = \sqrt{\left(\frac{\xi_X}{\Phi}\right)^2 + \left(\frac{\xi_Y}{\Psi}\right)^2}$$

Defining state variables $Y_1 = Y$, $Y_2 = \dot{Y}$, $Y_3 = \psi$ and $Y_4 = \dot{\psi}$, the following fourth-order autonomous dynamic system can be obtained,

$$\begin{aligned} \dot{Y}_1 &= Y_2,\ \dot{Y}_2 = -\frac{2k_1}{m}Y_1 - \frac{2}{m}F_Y - \frac{1}{m}F_T(Y_1), \\ \dot{Y}_3 &= Y_4,\ \dot{Y}_4 = -\frac{2k_2 d_1^2}{I}Y_3 - \frac{2a}{I}F_X \end{aligned} \tag{9}$$

Initial condition is set to $Y(0) = [0.0091, 0, 0, 0]^T$. Note that $Y_1(0) = \delta$, the dead band. The computed Lyapunov exponents are given in Table 3 (where S_0 is the spectral intensity so that the variance of the random variable is $2\pi S_0$) for selected cases of, (1) deterministic forward speed and dead band; (2) random forward speed but deterministic dead band; and (3) random dead band but deterministic forward speed.

Information dimensions are shown in Figure 2 and Table 4, and Figure 3 and Table 5 for the cases of randomness in forward speed and in dead band, respectively. It should be moentioned that Table 4 (or 5) is not simply a repetition of Figure 2 (or 3). Figure 2 or 3 covers a speed range of 10 – 20 m/s with an interval of 1. Table 4 or 5, on the other hand, provides finer detail for lower forward speeds where the nature of wheelset response is not as consistent as in the higher speed range. The following remarks are in order. (1) With the presence of large randomness ($S_0 \geq 100$), information dimension is "constant", regardless of speed; (2) The presence of small randomness ($S_0 \leq 1$) in speed may not alter the response from non-chaotic to chaotic, or vice versa. For example, the response goes from chaotic to non-chaotic when v = 10 – 10.8 m/s and 12 – 13.4 m/s, but remains non-chaotic when v = 11 – 12 m/s (see Figure 2 and Table 4); (3) The effect of small randomness in dead band is similar to that of randomness in forward speed; In particular, the response remains non-chaotic when v = 10.8 – 12.4 m/s (Figure 3 and Table 5); (4) With higher speed (v > 14 – 15 m/s) and higher S_0, the response is chaotic, regardless of S_0; and (5) There is a significant increase in information dimension when S_0 is increased from 1 to 100.

Information dimensions for the case of randomness in both the forward speed and dead band are listed in Table 6. It seems that for small randomness, the behaviour of the wheelset is influenced more by the randomness in dead band. For instance, information dimensions are consistently about 1.0 when S_0 = 1. Otherwise the randomness in forward speed plays the determining role in the dynamic behavior of the wheelset.

Table 3 Lyapunov exponents of the single-axle wheelset

deterministic forward speed and dead band				
v	λ_1	λ_2	λ_3	λ_4
10	8.90E-01	3.14E+00	-3.05E+02	-3.06E+02
11	-2.34E-01	-5.99E+00	-2.64E+02	-2.81E+02
12	-1.32E+00	-2.42E+01	-2.37E+02	-2.43E+02
13	2.33E+00	-1.38E+01	-2.28E+02	-2.27E+02
14	4.36E+00	-5.99E+00	-2.16E+02	-2.16E+02
15	6.04E-01	-3.55E+00	-1.96E+02	-2.04E+02
16	2.04E+00	-9.20E+00	-1.84E+02	-1.87E+02
17	4.60E+00	-1.61E+01	-1.55E+02	-1.89E+02
18	1.59E+00	-2.42E+00	-1.65E+02	-1.70E+02
19	3.16E+00	-5.53E+00	-1.55E+02	-1.60E+02
20	2.78E+00	-1.09E+01	-1.42E+02	-1.52E+02
random forward speed, S_0 = 10,000				
v	λ_1	λ_2	λ_3	λ_4
10	3.96E+01	2.70E+01	-2.79E+01	-3.90E+01
11	4.06E+01	2.64E+01	-2.87E+01	-4.12E+01
12	4.35E+01	2.79E+01	-3.25E+01	-4.67E+01
13	4.21E+01	2.77E+01	-3.12E+01	-4.34E+01
14	4.32E+01	2.84E+01	-3.16E+01	-4.38E+01
15	4.15E+01	2.76E+01	-3.06E+01	-4.24E+01
16	4.18E+01	2.78E+01	-2.99E+01	-4.22E+01
17	4.14E+01	2.76E+01	-2.88E+01	-4.03E+01
18	4.15E+01	2.80E+01	-2.97E+01	-4.16E+01
19	3.96E+01	2.59E+01	-2.85E+01	-3.99E+01
20	4.09E+01	2.63E+01	-2.97E+01	-4.22E+01
random dead band, S_0 = 100				
v	λ_1	λ_2	λ_3	λ_4
10	7.52E+00	-3.58E+00	-3.43E+01	-5.43E+01
11	8.03E+00	-2.23E+00	-3.09E+01	-5.05E+01
12	9.79E+00	-1.52E+00	-2.82E+01	-4.85E+01
13	1.00E+01	-1.32E+00	-2.52E+01	-4.58E+01
14	1.11E+01	-8.04E-01	-2.40E+01	-4.38E+01
15	1.08E+01	2.07E-01	-2.23E+01	-4.14E+01
16	1.17E+01	1.12E+00	-2.13E+01	-4.06E+01
17	1.23E+01	6.20E-01	-1.96E+01	-3.87E+01
18	1.28E+01	1.21E+00	-1.89E+01	-3.76E+01
19	1.31E+01	1.70E+00	-1.82E+01	-3.64E+01
20	1.36E+01	1.99E+00	-1.75E+01	-3.57E+01

It is noted that, in the computation of the information dimensions, the fixed step 4-th order Runge-Kutta method is used to numerically integrate the responses $Y(t_k)$ and variations Γ_i with a time step size of 0.001s over 15000 steps. The large integer M of Equ (3) is set to 7500. The (temporal) average of these 7500 values of λ_i is then considered the λ_i of a sample, or a realization. Finally, the ensemble averages of λ_i over 256 samples are used to determine information dimension by means of Equ (4).

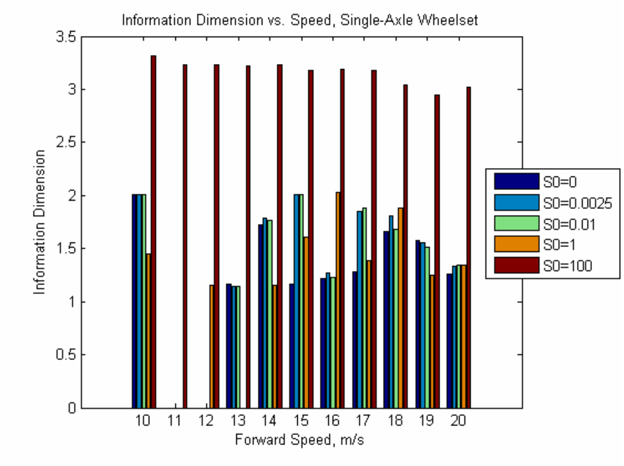

Figure 2 Information dimension versus mean forward speed with randomness in speed

Table 4 Information dimensions of the single-axle wheelset with randomness in forward speed

v	spectral intensity S_0					
(m/s)	0	0.0025	0.01	1.0	100.0	900.0
10.0	2.01	2.01	2.01	0.00	3.24	3.72
10.2	2.02	2.03	2.03	0.00	3.25	3.73
10.4	2.02	2.03	2.03	0.00	3.18	3.74
10.6	1.27	0.00	0.00	0.00	3.16	3.72
10.8	2.00	1.57	1.66	0.00	3.09	3.76
11.0	0.00	0.00	0.00	2.01	3.07	3.76
11.2	0.00	0.00	0.00	2.01	3.00	3.77
11.4	0.00	0.00	0.00	2.01	2.98	3.73
11.6	0.00	0.00	0.00	2.01	3.06	3.70
11.8	0.00	0.00	0.00	1.30	3.13	3.72
12.0	0.00	0.00	0.00	0.00	3.12	3.71
12.2	0.00	0.00	0.00	0.00	3.16	3.76
12.4	1.04	0.00	1.01	0.00	3.21	3.77
12.6	1.00	1.04	1.04	0.00	3.13	3.70
12.8	1.05	1.08	1.09	0.00	3.13	3.71
13.0	1.17	1.14	1.12	0.00	3.10	3.67
13.2	1.09	1.10	1.07	0.00	3.10	3.73
13.4	1.16	1.11	1.11	0.00	3.04	3.75
13.6	1.25	1.18	1.24	1.03	3.05	3.77
13.8	1.46	1.29	1.40	1.07	3.09	3.80
14.0	1.73	1.83	1.77	1.15	3.10	3.73
14.2	2.01	2.03	2.03	1.09	3.12	3.73
14.4	2.04	2.04	2.03	1.10	3.11	3.79
14.6	2.01	2.03	2.03	1.22	3.16	3.81
14.8	2.03	2.03	2.03	1.40	3.11	3.75
15.0	1.17	2.01	2.02	1.86	3.10	3.79

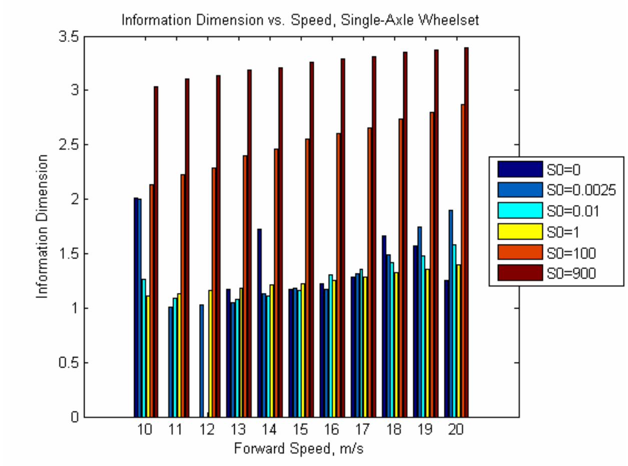

Figure 3 Information dimension versus forward speed with randomness in dead band

Table 5 Information dimensions of the single-axle wheelset with randomness in dead band

v	spectral intensity S_0					
(m/s)	0	0.0025	0.01	1.0	100.0	900.0
10.0	2.01	2.00	1.26	1.09	2.15	3.00
10.2	2.02	2.00	1.27	1.09	2.16	3.02
10.4	2.02	0.00	1.23	1.10	2.17	3.03
10.6	1.27	0.00	1.11	1.09	2.19	3.04
10.8	2.00	0.00	1.06	1.09	2.21	3.04
11.0	0.00	0.00	0.00	1.09	2.23	3.04
11.2	0.00	0.00	1.00	1.10	2.25	3.06
11.4	0.00	0.00	0.00	1.10	2.25	3.06
11.6	0.00	1.02	0.00	1.10	2.27	3.07
11.8	0.00	0.00	0.00	1.10	2.29	3.08
12.0	0.00	1.01	0.00	1.11	2.29	3.08
12.2	0.00	1.01	0.00	1.12	2.32	3.12
12.4	1.04	1.06	0.00	1.11	2.34	3.12
12.6	1.00	1.06	1.05	1.13	2.34	3.13
12.8	1.05	1.09	1.10	1.13	2.35	3.15
13.0	1.17	1.11	1.10	1.12	2.38	3.13
13.2	1.09	1.11	1.10	1.13	2.40	3.13
13.4	1.16	1.12	1.10	1.14	2.42	3.15
13.6	1.25	1.10	1.15	1.14	2.43	3.16
13.8	1.46	1.07	1.21	1.15	2.43	3.17
14.0	1.73	1.06	1.20	1.15	2.45	3.18
14.2	2.01	1.06	1.23	1.15	2.46	3.19
14.4	2.04	1.13	1.25	1.16	2.49	3.19
14.6	2.01	1.13	1.25	1.16	2.49	3.19
14.8	2.03	1.14	1.26	1.17	2.51	3.20
15.0	1.17	1.17	1.25	1.16	2.53	3.21

Table 6 Information dimensions of the single-axle wheelset with randomness in both forward speed and dead band

v (m/s)	spectral intensity S_0					
	0	0.0025	0.01	1.0	100.0	900.0
10.0	2.01	0.00	0.00	1.03	3.21	3.78
10.2	2.02	2.01	0.00	1.04	3.20	3.78
10.4	2.02	2.00	0.00	1.03	3.20	3.80
10.6	1.27	2.00	0.00	1.04	3.17	3.84
10.8	2.00	1.25	1.02	1.05	3.14	3.79
11.0	0.00	1.04	1.04	1.04	3.10	3.77
11.2	0.00	0.00	1.08	1.04	3.14	3.81
11.4	0.00	1.01	1.14	1.05	3.18	3.80
11.6	0.00	0.00	1.16	1.05	3.23	3.80
11.8	0.00	0.00	1.09	1.05	3.24	3.82
12.0	0.00	0.00	1.11	1.05	3.15	3.81
12.2	0.00	0.00	1.13	1.05	3.18	3.70
12.4	1.04	1.03	1.15	1.05	3.17	3.74
12.6	1.00	1.01	1.13	1.06	3.18	3.80
12.8	1.05	0.00	1.11	1.05	3.15	3.73
13.0	1.17	0.00	1.15	1.06	3.20	3.78
13.2	1.09	0.00	1.18	1.06	3.20	3.76
13.4	1.16	0.00	1.25	1.06	3.20	3.77
13.6	1.25	0.00	1.19	1.06	3.15	3.77
13.8	1.46	0.00	1.24	1.07	3.20	3.78
14.0	1.73	0.00	1.19	1.07	3.25	3.73
14.2	2.01	0.00	1.09	1.07	3.18	3.75
14.4	2.04	0.00	1.04	1.07	3.16	3.73
14.6	2.01	1.06	1.01	1.08	3.19	3.74
14.8	2.03	1.01	1.06	1.07	3.25	3.74
15.0	1.17	1.01	1.09	1.08	3.18	3.76

5 Two-axle wheelsets

The wheelset consists of four wheels on two axles (see Figure 4 and Table 7). The equations of the motion are [3],

$$\dot{y}_i = f_i(y_1, \cdots, y_{14}),\, i = 1,\ \cdots,\ 14 \tag{10}$$

where,

$$\begin{aligned}
f_i &= y_{i+7}\,,\ i = 1,\cdots,7 \\
f_8 &= -\left[A_1 + 2f_{xf} + F_T(y_1)\right]\big/m_\omega\,, \\
f_9 &= -\left(A_3 + 2af_{yf}\right)\big/I_{\omega y}\,, \\
f_{10} &= -\left[A_2 + 2f_{xr} + F_T(y_3)\right]\big/m_\omega\,, \\
f_{11} &= -\left(A_4 + 2af_{yr}\right)\big/I_{\omega y}\,, \\
f_{12} &= -(-A_1 - A_2 + A_5)\big/m_f\,, \\
f_{13} &= -(-bA_1 + bA_2 - A_3 - A_4 + A_6)\big/I_{fy}\,, \\
f_{14} &= -(-h_1A_1 - h_1A_2 + h_2A_5 + A_7)\big/I_{fr}\,.
\end{aligned} \tag{11}$$

and the A_i terms are,

$$\begin{aligned}
A_1 &= 2k_1(y_1 - y_5 - by_6 - h_1y_7)\,, \\
A_2 &= 2k_1(y_3 - y_5 - by_6 - h_1y_7)\,, \\
A_3 &= 2k_2d_1^2(y_2 - y_6)\,, \\
A_4 &= 2k_2d_1^2(y_4 - y_6)\,, \\
A_5 &= 2D_2(y_{12} - h_2y_{14}) + 2k_4(y_5 - h_2y_7)\,, \\
A_6 &= k_6y_6\,, \\
A_7 &= 2D_1d_2^2y_{14} + 2k_5d_2^2y_7 + 4k_3d_1^2y_7\,.
\end{aligned} \tag{12}$$

In Equ (11), F_T is the flange force that can be determined by Equ (7). The creep forces f_{xf} , f_{yf} , f_{xr} and f_{yr} are of the form,

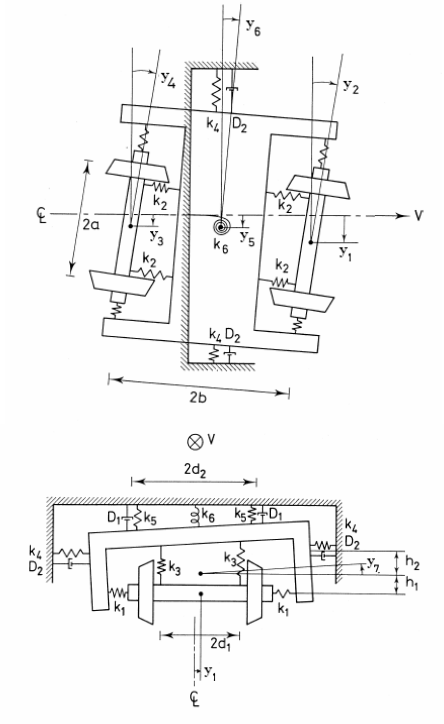

Figure 4 A two-axle wheelset

Table 7 Parametric values of the two-axle wheelset

$a = 0.716$ m	$k_1 = 0.1823\times10^7$ N/m
$b = 1.074$ m	$k_2 = 0.3646\times10^7$ N/m
$d_1 = 0.62$ m	$k_3 = 0.3646\times10^7$ N/m
$d_2 = 0.68$ m	$k_4 = 0.1823\times10^6$ N/m
$D_1 = 0.2\times10^5$ N-s/m	$k_5 = 0.3333\times10^6$ N/m
$D_2 = 0.292\times10^5$ N-s/m	$k_6 = 0.271\times10^7$ N/m
$\delta = 0.0091$ m	$\lambda = 0.05$
$G\pi a_e b_e = 0.6563\times10^7$ N	$\mu N = 10000$ N
$h_1 = 0.0762$ m	$m_f = 2918$ kg
$h_2 = 0.6584$ m	$m_\omega = 1022$ kg
$I_{fy} = 6780$ kg-m^2	$\Phi = 0.60252$
$I_{fr} = 6780$ kg-m^2	$\Psi = 0.54219$
$I_{\omega y} = 678$ kg-m^2	$r_0 = 0.4572$ m
$k_0 = 14.6\times10^6$ N/m	

$$F_{xf} = \frac{\xi_{xf}}{\Psi}\frac{F_{Rf}}{\xi_{Rf}}, \quad F_{xr} = \frac{\xi_{xr}}{\Psi}\frac{F_{Rr}}{\xi_{Rr}}, \qquad (13)$$
$$F_{yf} = \frac{\xi_{yf}}{\Phi}\frac{F_{Rf}}{\xi_{Rf}}, \quad F_{yr} = \frac{\xi_{yr}}{\Phi}\frac{F_{Rr}}{\xi_{Rr}}$$

in which the creepages are

$$\xi_{xf} = \frac{y_8}{v} - y_2\,,\ \xi_{yf} = \frac{ay_9}{v} + \frac{\lambda y_1}{r_0}, \qquad (14)$$
$$\xi_{xr} = \frac{y_{10}}{v} - y_4\,,\xi_{yr} = \frac{ay_{11}}{v} + \frac{\lambda y_3}{r_0}$$

such that the resultant creepages are

$$\xi_{Rf} = \sqrt{\left(\frac{\xi_{xf}}{\Psi}\right)^2 + \left(\frac{\xi_{yf}}{\Phi}\right)^2}, \qquad (15)$$
$$\xi_{Rr} = \sqrt{\left(\frac{\xi_{xr}}{\Psi}\right)^2 + \left(\frac{\xi_{yr}}{\Phi}\right)^2}$$

and the resultant creepage forces are, with $u_f = \frac{G\pi a_e b_e}{\mu N}\xi_{Rf}$ and $u_r = \frac{G\pi a_e b_e}{\mu N}\xi_{Rr}$,

$$F_{Rf} = (\mu N)\begin{cases} u_f - \frac{1}{3}{u_f}^2 + \frac{1}{27}{u_f}^3 & u_f < 3 \\ 1 & u_f \geq 3 \end{cases} \qquad (15)$$
$$F_{Rr} = (\mu N)\begin{cases} u_r - \frac{1}{3}{u_r}^2 + \frac{1}{27}{u_r}^3 & u_r < 3 \\ 1 & u_r \geq 3 \end{cases}$$

The initial condition is $Y(0) = [0.0091, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$. Note that $Y_1(0) = \delta$. Similar to the computations performed on the single-axle wheelset, the fixed step 4-th order Runge-Kutta method is used. However, a smaller time step size of 0.0001s is used over 25000 steps, the latter half of which is used for the determination of λ_i by Equ (3). Then the temporal average of these 12500 λ_i values is used for the determination of the ensemble averages of λ_i over 256 samples. The two-axle wheelset system requires a much longer computing time, largely due to the larger number of state variables (14 versus 4), and due to the employment of Graham-Schmidt scheme in the computation of Lyapunov exponents which drastically expands the size of state-variable vector (210 versus 20). In Figure 5, the computed information dimensions are shown for the case of randomness in forward speed, with the mean forward speed varying between 100 and 200 m/s.

Unlike the single-axle wheelsets which are more susceptible to the presence or absence of randomness in the system (with small randomness in particular), the two-axle wheelsets exhibit a "consistent" behavior, being chaotic. It is interesting to note that, at higher mean forward speed ($v \geq 170$ m/s), as the spectral intensity S_0 increases, the information dimension increases first, then drops, and increases again (see Figure 5, top half). For large S_0 (Figure 5, bottom half), information dimension becomes almost constant across the entire 100 – 200 m/s speed range.

When randomness in dead band is incorporated (Figure 6), the information dimensions are found to be quite constant over the entire speed range, and to be increasing as S_0 increases. Interestingly, when the randomness is small ($S_0 < 1$, see Figure 6, top half, and Table 8 with smaller interval between 100 and 120 m/s) the information dimensions are found to be less than those of the deterministic cases.

6 Conclusions

This paper presents a numerical simulation that aims at investigating the effect of randomness in forward speed, in dead band, and in both forward speed and dead band, on the dynamic behaviors of single- and two-axle railway wheelsets. Pseudo-random numbers are incorporated into the dynamic models of the wheelsets. Lyapunov exponents and information dimension are determined. It is found that the introduction of small to moderate randomness does not necessarily trigger chaotic response. In fact, the contrary may happen. That is, the presence of small randomness may suppress chaotic response otherwise in existence. Large amount of randomness, on the other hand, seems to universally push the response into the chaotic realm. With respect to the single-axle wheelsets, when randomness is present in both the forward speed and dead band, the latter (that is, randomness in dead band) seems to play a more defining role in the behavior of the wheelset as long as randomness is small ($S_0 \leq 1$). Beyond that, the wheelsets behave as if randomness in dead band were not present.

It should also be mentioned that forward speed of the single-axle wheelsets is chosen to vary between 10 and 20 m/s because Ref. [1] found the wheelset to experience, within that speed range, a variety of complex dynamical behaviors such as supercritical Hopf bifurcation, flange contacts, chaos explosions and reverse period doubling. As to the two-axle wheelsets, Ref [3] determined, within the speed range of 100 – 200 m/s, the existence of four saddle points, and points of pitchfork and Hopf bifurcations.

In regard to CPU time, on an IBM A31, the single- and two-axle wheelset systems take 148.54 and 5920.8 seconds, respectively, to complete one computational run that includes pseudo-random number generation, Lyapunov exponents and information computation, and ensemble averaging over 256 samples.

7 References

[1] Y. Nath and K. Jayadev, "Influence of yaw stiffness on the nonlinear dynamics of railway wheelset," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, pp.179-190, 2005.

[2] P.J. Vermeulen and K.L. Johnson, “Contact of non-spherical elastic bodies transmitting tangential forces,” *Journal of Applied Mechanics*, vol. 31, pp.339-340, 1964.

[3] C.N. Jensen, M. Golubitsky and H. True, “Symmetry, generic bifurcations, and mode interaction in nonlinear railway dynamics,” *International Journal of Bifurcation and Chaos*, vol. 9, pp.1321-1331, 1999.

[4] A. Wolf, J.B. Swift, H.L. Swiney and J.A. Vasano, “Determining Lyapunov exponents from a time series,” *Physica*, vol 16D, pp.285-317, 1985.

[5] C.W.S. To and M.L. Liu, “Lyapunov exponents and information dimensions of multi-degree-of-freedom systems under deterministic and stationary random excitations,” in *IUTAM Symposium on Advances in Nonlinear Stochastic Mechanics*, A. Naess and S. Krenk (Eds.), pp.449-458, 1995.

Table 8 Information dimensions of the two-axle wheelset with randomness in dead band

v	spectral intensity S_0				
(m/s)	0	2.5^{-3}	0.01	1.0	100.0
100	3.12	2.01	2.11	7.18	10.32
101	3.11	2.02	2.10	7.21	10.32
102	3.12	2.01	2.16	7.27	10.34
103	3.13	2.05	2.09	7.28	10.34
104	3.14	2.07	2.11	7.31	10.40
105	3.15	2.08	2.14	7.36	10.37
106	3.16	2.08	2.17	7.32	10.38
107	3.16	2.09	2.17	7.46	10.38
108	3.16	2.12	2.15	7.47	10.41
109	3.17	2.14	2.13	7.49	10.41
110	3.17	2.13	2.20	7.56	10.39
111	3.17	2.16	2.23	7.59	10.42
112	3.17	2.17	2.21	7.62	10.42
113	3.17	2.18	2.21	7.66	10.44
114	3.18	2.18	2.20	7.69	10.47
115	3.18	2.19	2.26	7.68	10.47
116	3.18	2.19	2.28	7.75	10.48
117	3.19	2.22	2.28	7.73	10.47
118	3.19	2.23	2.25	7.83	10.48
119	3.19	2.22	2.28	7.89	10.48
120	3.20	2.22	2.32	7.85	10.52
130	3.22	2.37	2.52	8.10	10.58
140	3.21	2.49	2.76	8.37	10.63
150	3.26	2.62	2.99	8.61	10.68
160	3.31	2.73	3.25	8.84	10.73
170	3.39	2.92	3.52	8.98	10.74
180	3.57	3.17	3.95	9.07	10.80
190	3.94	3.48	4.18	9.22	10.85
200	4.19	4.12	4.57	9.31	10.85

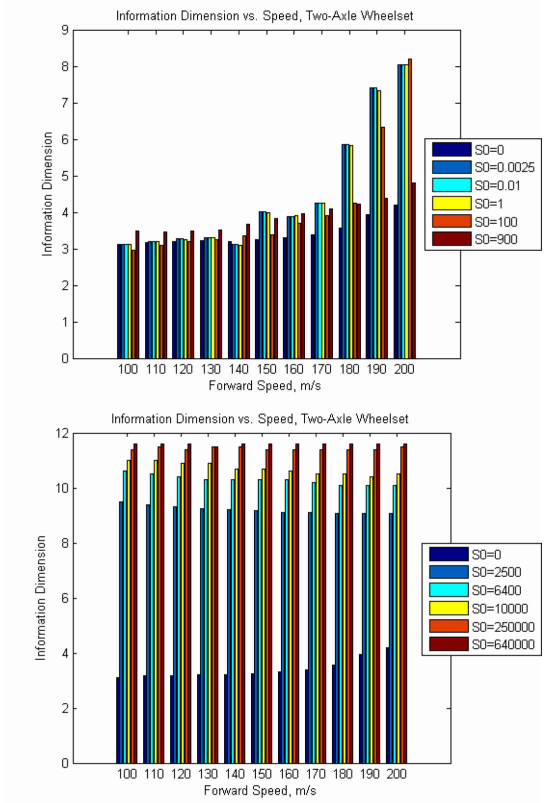

Figure 5 Information dimension versus mean forward speed with randomness in speed

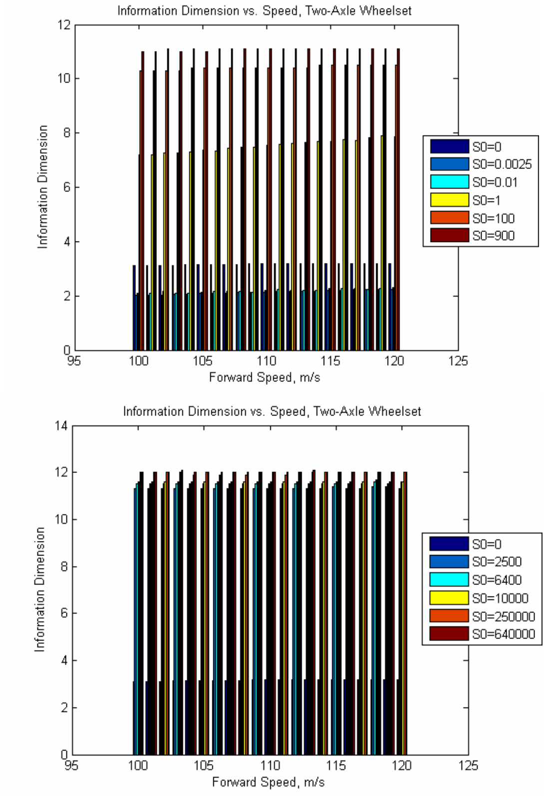

Figure 6 Information dimension versus forward speed with randomness in dead band