On solving equations of algebraic sum of equal powers

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Abstract *It is well known that a system of equations of sum of equal powers can be converted to an algebraic equation of higher degree via Newton's identities. This is the Viete-Newton theorem. This work reports generalizations of the Viete-Newton theorem to a system of equations of algebraic sum of equal powers. By exploiting some facts from algebra and combinatorics, it is shown that a system of equations of algebraic sum of equal powers can be converted in a closed form to two algebraic equations, whose degree sum equals to the number of unknowns of the system of equations of algebraic sum of equal powers.*

Keywords: Algebraic sum of equal powers, Newton's identities, System of equations, Roots of a polynomial

1 Introduction

Let $[n] := \{1, 2, \ldots, n\}$ and

$$\prod_{\nu=1}^{n}(x - x_\nu) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0. \tag{1.1}$$

It is well known that the elementary symmetric polynomial

$$\sigma_k(x_i \mid i \in [n]) := \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}, \quad k \in [n],$$

whose variables are all roots $x_1, x_2, \ldots, x_n$ of the algebraic equation (1.1) of degree n assumes values $a_k,\ k \in [n]$. This is the well-known Viete's theorem. By exploiting Newton's identities, we can represent the elementary symmetric polynomials by the symmetric polynomials of sum of the equal powers

$$s_k(x_i \mid i \in [n]) := \sum_{i=1}^{n} x_i^k, \quad k \ge 1.$$

This can be applied to solve a system of equations of sum of equal powers

$$\begin{cases} x_1 + x_2 + \cdots + x_n = p_1, \\ x_1^2 + x_2^2 + \cdots + x_n^2 = p_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^n + x_2^n + \cdots + x_n^n = p_n. \end{cases} \tag{1.2}$$

We now reformulate it as the following theorem.

Viete-Newton Theorem The solution $x_1, x_2, \ldots, x_n$ of the nonlinear system of equations (1.2) amounts to the all solutions of the algebraic equation (1.1) of degree n, where the coefficients $a_1, a_2, \ldots, a_n$ of the equation (1.1) are determined by

$$a_\nu := \frac{1}{\nu}\sum_{\mu=1}^{\nu}(-1)^{\mu-1}p_\mu a_{\nu-\mu},\ \nu \in [n];\ a_0 := 1. \tag{1.3}$$

Consider now the following system of equations of algebraic sum of equal powers

$$\left\{\begin{array}{llllllll} x_1 & +x_2 & +\cdots & +x_k & -x_{k+1} & -x_{k+2} & -\cdots & -x_n & = p_1, \\ x_1^2 & +x_2^2 & +\cdots & +x_k^2 & -x_{k+1}^2 & -x_{k+2}^2 & -\cdots & -x_n^2 & = p_2, \\ \multicolumn{9}{c}{\dotfill} \\ x_1^n & +x_2^n & +\cdots & +x_k^n & -x_{k+1}^n & -x_{k+2}^n & -\cdots & -x_n^n & = p_n. \end{array}\right. \tag{1.4}$$

Our results generalize the Viete-Newton theorem to the above nonlinear system of equations.

Systems of equations of algebraic sum of equal powers have wide applications in many areas such as computational algebra, communications, robotics, chemistry, mechanics [1] and mathematics mechanization [2]. Therefore, it deserves to investigate these systems. Our more direct motivations are to constructively solve an perfect spline interpolation problem [3]. It is well known that perfect splines are usually extremal functions satisfying some extremal conditions in approximation theory. This is indeed the case. The extremal function for our proposed new notion “the best quadrature for the Sobolev class $KW^r[a,b]$ based on given Hermite information” is no other than a kind of perfect spline interpolation functions. A system of equations which gives the free nodes of the interpolation perfect spline is just a system of equations algebraic sum of equal powers.

In present, the available and most widely known methods for solving the above nonlinear system are such as resultants, Gröbner bases, characteristic sets, Wu-Ritt process. These methods, for example, Wu-Ritt process, however, suffer from high computational complexity. By exploiting some properties of symmetric groups and using techniques from combinatorial analysis [4, 5], we first generalize the Viete-Newton theorem and then apply the generalized theorem to the system (1.4) to show that the system can be converted in a closed form to two algebraic equations, whose degree sum equals to the number of unknowns of the system of equations of algebraic sum of equal powers. This helps to greatly reduce computational complexity.

Finally, an example is included to illustrate the solving process.

2 Main results

Theorem 2.1. The solution $x_1, x_2, \ldots, x_k$ and $x_{k+1}, x_{k+2}, \ldots, x_n$ of the nonlinear system (1.4) amounts, when $k \geq n/2$, respectively to the solutions of the algebraic equation of degree k

$$\prod_{\nu=1}^{k}(x - x_\nu) = x^k - a_1 x^{k-1} + a_2 x^{k-2} - \cdots + (-1)^k a_k = 0 \tag{2.1}$$

and of algebraic equation of degree $n-k$

$$\prod_{\nu=k+1}^{n}(x - x_\nu) = x^{n-k} - a_{k+1} x^{n-k-1} + a_{k+2} x^{n-k-2} - \cdots + (-1)^{n-k} a_n = 0, \tag{2.2}$$

where $a_1, a_2, \ldots, a_n$ satisfy the following linear system of equations

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-k} \\ \vdots \\ b_k \\ b_{k+1} \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} -1 & & & & & 1 & & & \\ & -1 & & & & b_1 & 1 & & \\ & & \ddots & & & \vdots & \vdots & \ddots & \\ & & & -1 & & b_{n-k-1} & b_{n-k-2} & \cdots & 1 \\ & & & & \ddots & \vdots & \vdots & & \vdots \\ & & & & -1 & b_{k-1} & b_{k-2} & \cdots & b_{2k-n} \\ & & & & & b_k & b_{k-1} & \cdots & b_{2k-n+1} \\ & & & & & \vdots & \vdots & & \vdots \\ & & & & & b_{n-1} & b_{n-2} & \cdots & b_k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-k} \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{pmatrix} = 0. \tag{2.3}$$

Here we have put

$$b_\nu := \frac{1}{\nu} \sum_{\mu=1}^{\nu} (-1)^{\mu-1} p_\mu b_{\nu-\mu},\ \nu = 1, 2, \ldots, n;\ b_0 := 1. \tag{2.4}$$

If $k = n$, it is easily seen that Theorem 2.1 reproduces the Viete-Newton theorem since then $b_\nu = a_\nu,\ \nu \in [n]$.

Theorem 2.2. Suppose that $x_1, x_2, \dots, x_n$ satisfy the system (1.4) of algebraic equations of sum of equal powers. Let $k \geq n/2$ and define

$$p_\nu := x_1^\nu + x_2^\nu + \cdots + x_k^\nu - x_{k+1}^\nu - x_{k+2}^\nu - \cdots - x_n^\nu, \quad \nu = n+1, n+2, \dots. \tag{2.5}$$

Then we have

$$p_\nu = p_{\nu-1} b_1 - p_{\nu-2} b_2 + \cdots + (-1)^\nu p_1 b_{\nu-1} + (-1)^{\nu+1} \nu b_\nu, \quad \nu = n+1, n+2, \dots, \tag{2.6}$$

where,

$$b_\nu := -b_{\nu-1} a_{k+1} - b_{\nu-2} a_{k+2} - \cdots - b_{\nu+k-n} a_n, \quad \nu = n+1, n+2, \dots.$$

Here, $a_{k+1}, a_{k+2}, \dots, a_n$ is determined by the linear system (2.3).

Remark 2.1. It is interesting to note that the computation of p_ν, from Theorem 2.2, can be recursively proceeded and need not be resorted to solutions of (1.4) or of two equations (2.1) and (2.2) of high degree.

Example 1. Consider the following nonlinear system of equations

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = p_1, \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 = p_2, \\ x_1^3 + x_2^3 - x_3^3 - x_4^3 = p_3, \\ x_1^4 + x_2^4 - x_3^4 - x_4^4 = p_4. \end{cases} \tag{1.4a}$$

Its solution corresponds to the all solutions of the following two quadratic equations

$$x^2 - a_1 x + a_2 = 0, \tag{2.1a}$$

$$x^2 - a_3 x + a_4 = 0 \tag{2.2a}$$

via linear system of equations

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & b_1 & 1 \\ 0 & 0 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0. \tag{2.3a}$$

Here we have put

$$\begin{aligned} b_1 &= p_1, \\ b_2 &= \frac{1}{2}(p_1^2 - p_2), \\ b_3 &= \frac{1}{6}(p_1^3 - 3p_1 p_2 + 2p_3), \\ b_4 &= \frac{1}{24}(p_1^4 - 6p_1^2 p_2 + 3p_2^2 + 8p_1 p_3 - 6p_4). \end{aligned} \tag{2.4a}$$

If

$$b_2^2 - b_1 b_3 = \frac{1}{12}(p_1^4 - 4p_1 p_3 + 3p_2^2) \neq 0, \tag{2.5a}$$

we have that the linear system (2.3a)

$$\begin{aligned} a_3 &= \frac{b_1 b_4 - b_2 b_3}{b_2^2 - b_1 b_3}, \quad a_4 = \frac{b_3^2 - b_2 b_4}{b_2^2 - b_1 b_3}. \\ a_1 &= b_1 + a_3, \quad a_2 = b_2 + b_1 a_3 + a_4. \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - 4a_2}\right) = \frac{1}{2}\left(b_1 + a_3 \pm \sqrt{(b_1 - a_3)^2 - 4(b_2 + a_4)}\right), \\ x_{3,4} &= \frac{1}{2}\left(a_3 \pm \sqrt{a_3^2 - 4a_4}\right). \end{aligned}$$

If the condition (2.5a) is not satisfied, but $b_2 \neq 0$, then whether the nonlinear system (1.4a) has a solution depends on the following condition

$$b_3^2 - b_2 b_4 = 0 \tag{2.6a}$$

is satisfied or not. If this is the case, then the nonlinear system (1.4a) is undetermined, and thus it has an infinite number of solutions; Otherwise, the the system (1.4a) is inconsistent and thus has no solutions. If further, the condition (2.5a) is not satisfied but $b_2 = 0$. Under such a condition, if $b_1 = b_3 = b_4 = 0$, then the system (1.4a) is undetermined; otherwise it is inconsistent.

Remark 2.2. If k and $n-k$ are large, we can use parallel iteration [6] to find all roots of polynomial equations of degree k and $n-k$ simultaneously.

Example 2. In [3], we are to find a nonlinear quadrature and its worst case error bound. The problem is equivalent to finding p_5, given the system (1.4a). By Example 1 and Theorem 2.2, we have

$$p_5 = p_4 b_1 - p_3 b_2 + p_2 b_3 - p_1 b_4 + 5 b_5,$$

where

$$b_5 = -b_4 a_3 - b_3 a_4.$$

Putting all their expressions from Example 1 yields

$$\begin{aligned} p_5 \quad = \quad & \frac{1}{144(p_1^4 + 3p_2^2 - 4p_1 p_3)} \left(-p_1^9 - 18p_1^5 p_2^2 + 24p_1^6 p_3 - 360p_1^2 p_2^2 p_3 + 360p_1^3 p_2 p_4 \right. \\ & \left. +80(-4p_3^3 + 9p_2 p_3 p_4) + 135p_1(p_2^4 - 4p_4^2)\right). \end{aligned}$$

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