Biorthogonal Sampling Functions Associated With Meyer Type Wavelets

Xiaoping Shen
shen@math.ohiou.edu
Department of Mathematics
Ohio University
Athens, OH 45701, U.S.A.

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Abstract

In this article, we study a class of biorthogonal sampling functions in the context of bandlimited wavelets, Meyer type wavelets. Originally raised in the construction of bandlimited wavelets, these sampling functions also possess a similar structure to the scaling functions of wavelets with adjustable bandwidth parameters. In addition, these sampling functions are infinite impulse response (IIR) filters and share all the principal advantage that the IIR type has such as computational efficiency. They are easy to compute with fast decreasing property in time domain and suitable for representing bandlimited signals with sharp cut-off.

Numerical examples are given to illustrate the construction of sampling functions and properties of associated sampling series.

Keyword: Sampling functions, bandlimited functions, Meyer wavelets.

1. Introduction

Sampling is a process of converting a signal (e.g., a function of continuous time) into a numeric sequence (a function of discrete time). The process is also called analog-to-digital conversion, or simply digitizing. More precisely, analog-to-digital conversion actually consists of the combination of two processes: sampling, which involves converting the domain of the signal from continuous-time to discrete-time, and quantization, which involves converting the signal samples to analog signal. The Whittaker-Shannon-Kotel'nikov sampling theorem is a fundamental theorem in this field, which is known in communication and electrical engineering literature as the Shannon Sampling Theorem [7],

Whittaker-Shannon-Kotel'nikov Sampling Theorem. If $f(t)$ is a signal bandlimited to $[-\sigma, \sigma]$, i. e.,

$$f(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} F(w) dw$$

for some function $F \in L^2(-\sigma, \sigma)$, then $f(t)$ has the series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{\sigma}) \frac{\sin(\sigma t - n\pi)}{\sigma t - n\pi}.$$

The classic Shannon sampling theorem plays a very important role in modern communication theory, by which a band-limited signal is fully recovered from its (infinitely many) discrete samples. It is also close related to the modern wavelet theory. Unfortunately, the Shannon sampling sequence surfers from its slow convergence and the truncation error analysis is critical. This problem is usually handled by introducing convergence acceleration factors/regularizer (similar to the regularization ideas used to the treatment of slow convergence or non convergence series). Many extensions of Shannon's sampling theorem can be found in literature (say for example, [4], [9]). With the rapidly development of wavelet theory and its applications in signal and image processing, sampling theory has been discussed in context of wavelet analysis ([1], [6], and [10].

In this article, we study a class of biorthogonal sampling functions (see definition below) in connection to bandlimited wavelets, Meyer type wavelets. Raised originally in the construction of bandlimited wavelets, these sampling functions also possess a similar structure to the scaling functions of wavelets with adjustable bandwidth parameters. These sampling functions are infinite impulse response (IIR) filters and share all the principal advantage that the IIR type has such as computational efficiency, the IIR uses much fewer terms than a finite Impulse Response (FIR) filters. This is particularly imporportant to realize a filter with a sharp cut-off.

We assume that readers have fundamental knowledge in wavelet analysis. More details about wavelet theory can be find in [3] and [11]. To make the article self-contained, we present here a few elements of orthogonal wavelet theory, in which an orthonormal basis $\{\psi_{mn}\}$ of $L^2(R)$ is constructed having the form

$$\psi_{mn}(t) = 2^{m/2}\psi_{mn}(2^m t - n), \quad n, m \in R,$$

where $\psi(t)$ is the "mother wavelet". Usually it is not constructed directly but rather from another function called the "scaling function" $\phi(t) \in L^2(R)$. The scaling function ϕ, is chosen in such a way that,

(i) $\int_{-\infty}^{\infty} \phi(t)\phi(t-n)dt = \delta_{0,n}, \quad n \in Z,$

(ii) $\phi(t) = \sum_{k=-\infty}^{\infty} \sqrt{2} c_k \phi(2t-k),$
where $\{c_k\}_{k\in Z} \in l^2,$

(iii) For each $f \in L^2(R)$, $\varepsilon > 0$, there is a function $f_m(t) = \sum_n a_{mn}\phi(2^m t - n)$ such that $\|f_m - f\| < \varepsilon$,where $a_{mn} = \int_{-\infty}^{\infty} f(t)\phi(2^m t - n)dt$.

These conditions lead to a "multiresolution approximation" $\{V_m\}_{m\in Z}$, consisting of closed subspaces of $L^2(R)$. The space V_m is taken to be the closed linear span of $\{\phi(2^m t - n)\}_{n\in Z}$. Because of (ii), the V_m are nested, i.e. $V_m \subseteq V_{m+1}$, $m = -\infty, ..., \infty$ and because of (iii), $\cup_{m=-\infty}^{\infty} V_m$ is dense in $L^2(R)$.

2. Biorthogonal Sampling Functions and Their Properties

In this section, we begin with a couple of definitions in sampling theory, and give a brief reviewing of Meyer wavelets and their sampling properties.

There many wavelet bases have constructed and employed for different purposes. In this work, the scaling function of the Meyer wavelet is used to construct a wavelet basis. Recall the construction of Meyer wavelets. Let h be a probability density function with support in [-$\frac{\pi}{3}$, $\frac{\pi}{3}$] and define $\phi(t)$ as the function whose Fourier transform is the non negative square root of the integral

$$\widehat{\phi}(w) = \left(\int_{w-\pi}^{w+\pi} h(u)du\right)^{\frac{1}{2}}, \qquad (1)$$

then $\widehat{\phi}$ has support in [-$\frac{4\pi}{3}$, $\frac{4\pi}{3}$], and $\widehat{\phi} = 1$, for $w \in$ [-$\frac{2\pi}{3}$, $\frac{2\pi}{3}$]. The mth scaling space V_m is composed of $2^{m+2}\pi/3$ band limited functions. It has good frequency localization but relatively poor time localization. In addition, the Fourier transform of the mother wavelet vanishes in a neighborhood of the origin. We summarize some related properties,

P1. If $\phi(t)$ is a scaling function of Meyer wavelet, then

(i) ϕ is bandlimited function with

$$\begin{aligned} &\text{support } \{|\widehat{\phi}(w)|^2\} \\ &= \quad [-\pi-\varepsilon, \pi+\varepsilon] \subseteq [-\frac{4\pi}{3}, \frac{4\pi}{3}], \end{aligned}$$

(ii) $|\widehat{\phi}(w)| = 1$, for $|w| \le \frac{2\pi}{3}$,

(iii) The orthogonality condition is equivalent to,

$$\sum_{k=-\infty}^{\infty} |\widehat{\phi}(w+2k\pi)|^2 = \int_{-\infty}^{\infty} h(u)du = 1.$$

Unfortunately, the scaling function of a Meyer type wavelet is not a sampling function. However, we have

P2. Let ϕ be a real, symmetric scaling function satisfying (1) with $h \in C^r$, $r > 1$, such that

$$\widehat{\phi}^*(w) := \sum_{n=-\infty}^{\infty} \phi(n)e^{-iwn} \neq 0,\ w \in \Re. \quad (2)$$

If we define function $S(t)$ by its Fourier transform,

$$\widehat{S}(w) = \frac{\widehat{\phi}(w)}{\widehat{\phi}^*(w)}, \quad (3)$$

then $\{S(t-n)\}_{n=-\infty}^{\infty}$ is a biorthogonal Riesz basis of V_0.

P3. Let S be as in Property 2 and $h \in C^r(R)$, then $\widehat{S}(w)$ is an even function and satisfies the following properties:

(i) $\widehat{S}(w) = \frac{\widehat{\phi}(w)}{\widehat{\phi}(w-2\pi)+\widehat{\phi}(w)+\widehat{\phi}(w+2\pi)}$.

(ii) $S(n) = \delta(n)$, where $\delta(n)$ is the Kronecker Delta, $\delta(n) = 1$,when $n = 0$, $\delta(n) = 0$, when $n \neq 0$.

(iii) $\widehat{S}(2n\pi) = \delta(n)$, n is any integer, $\widehat{S}^{(n)}(0) = 0, 1 \le n \le r$.

Now we recall definition for sampling functions.

Definition 1 *(Sampling function) $f \in C(R)$ is a sampling function if*

$$f(k) = \delta(k),\ k \in Z.$$

Notice that, for $f \in V_0, S$ defined by (3), then

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n).$$

Example 2 *(Shannon sampling function) The Shannon sampling function ϕ is the famous sinc function defined as*

$$sinc(t) = \frac{\sin \pi t}{\pi t}, \quad (4)$$

and the associated mother wavelet is given by

$$\psi(t) = \frac{\sin \pi(t-\frac{1}{2}) - \sin 2\pi(t-\frac{1}{2})}{\pi(t-\frac{1}{2})}$$

The scaling function of the Shannon wavelet (4) is the only one among all the standard orthogonal scaling functions that satisfies translation invariant properties. Figure 1 shows the scaling function and the mother wavelet for Shannon system. A plot indicating how sinc functions sum together to reconstruct a discrete signals is shown in Figure 2. The figure shows a superposition of five sinc functions, each at unit amplitude, and displaced by one-sample intervals.

$$x = \{..., 0, 1, 1, 1, 1, 1, 0, ...\}, \quad (5)$$

or equivalently,

$$x(t) = \sum_{k=-2}^{2} \delta(t-k).$$

3. Examples

In this section, we give two interesting examples. In first example, we consider the Raised-cosine sampling functions which related to the Raised-cosine wavelet [13]. In the second example, we consider a group of sampling function, constructed by using the generate function h of the scaling function (1) and (3).

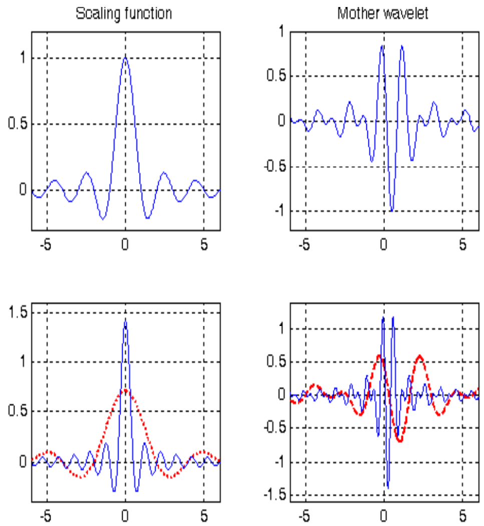

Figure 1: The Shannon wavelet system. The scaling function (Left of top row) and the associated mother wavelet (Right of top row). The scaling function (Left bottom row) and the mother wavelet (Right bottom row) at level $m = -1$ and $m = 1$, respectively.

3.1 The Raised-cosine Sampling function

In this example, we construct the Raised-cosine sampling function which is related to the Raised cosine wavelet. The scaling function for the Raised-cosine wavelet is the square root of raised cosine function which is given by

$$\phi(t) = \frac{1}{\pi t[1-(4\beta t)^2]}[\sin\pi(1-\beta)t - 4\beta t\cos\pi(1+\beta)t] \tag{6}$$

The associated mother wavelet is defined as

$$\begin{aligned}\psi(t+\frac{1}{2}) = {} & \frac{1}{\pi t[(4\beta t)^2-1]}[\sin\pi(1+\beta)t - 4\beta t\cos\pi(1-\beta)t] \\ & -\frac{1}{\pi t[(8\beta t)^2-1]}[\sin 2\pi(1-\beta)t \\ & +8\beta t\cos 2\pi(1+\beta)t],\end{aligned} \tag{7}$$

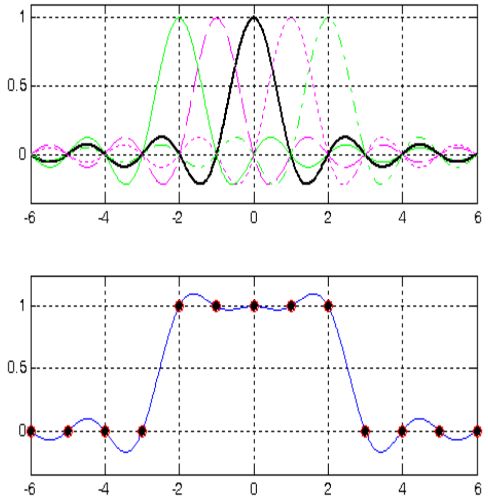

Figure 2: Shannon sampling function and its shiftings (top) and the discrete sequence recovered by using 5 terms of the Shannon sampling series.

where $0 \leq \beta \leq \frac{1}{3}$. Notice that, $\phi(t)$ is the sinc function when $\beta = 0$.

The raised cosine wavelets are bandlimited, their Fourier transforms are given by

$$\widehat{\phi}(w) = \begin{cases} 1 & 0 \leq |w| \leq \pi(1-\beta), \\ \cos[\frac{|w|}{4\beta} - \frac{\pi(1-\beta)}{4\beta}], & \pi(1-\beta) \leq |w| \leq \pi(1+\beta), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

and

$$\widehat{\psi}(w) = e^{-i\frac{w}{2}}[\widehat{\phi}(w+2\pi) + \widehat{\phi}(w-2\pi)]\widehat{\phi}(\frac{w}{2})$$

$$= e^{-i\frac{w}{2}} \begin{cases} 0 & 0 \le |w| \le \pi(1-\beta), \\ \cos[\frac{\pi-|w|}{4\beta} + \frac{\pi}{4}], & \pi(1-\beta) \le |w| \le \pi(1+\beta), \\ 1, & \pi(1+\beta) \le |w| \le 2\pi(1-\beta), \\ \cos[\frac{|w|/2-\pi(1-\beta)}{4\beta}], & 2\pi(1-\beta) \le |w| \le 2\pi(1+\beta), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Figure 3 shows scaling function and associated mother wavelet in time and frequency domain. The Raised-cosine sampling function S corresponding with the scaling function ϕ belongs to the scaling space V_0. It can be constructed by using (3) which is given by its Fourier transform:

$$\widehat{S}(w) = \begin{cases} 1, & 0 \le |w| \le \frac{3\pi}{4}, \\ \frac{1}{2}(1-\tan|w|), & \frac{3\pi}{4} \le |w| \le \frac{5\pi}{4}, \\ 0, & \text{else.} \end{cases} \tag{10}$$

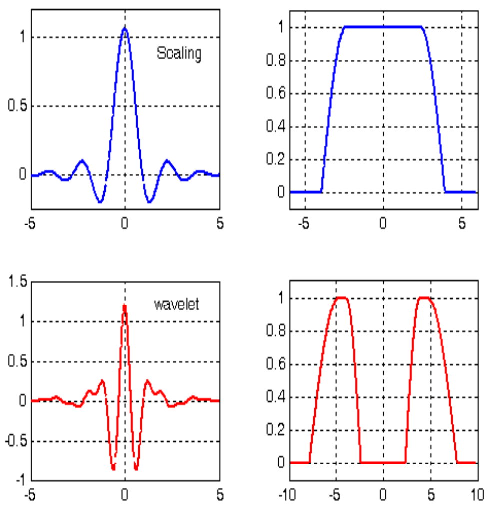

Figure 3: The scaling function (top row) and mother wavelet of raised cosine wavelets (bottom row) in time and frequency domain. $\beta = 1/4$.

A plot indicating how the sampling functions sum together to reconstruct the discrete signal (5) is shown in Figure 4. The figure shows a superposition of five raised cosine sampling functions, each at unit amplitude, and displaced by one-sample intervals.

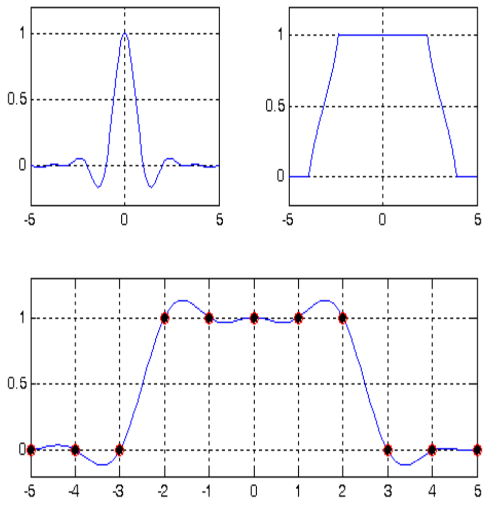

Figure 4: Raised-cosine sampling function in time and frequency domain (top) and the discrete sequence recovered by using 5 terms of the Raised-cosine sampling series.

Both Shannon sampling function and the Raised-cosnine sampling function have slow decreasing rate in time domain, and therefore, their corresponding sampling series convergence slowly.

3.2 Construct Sampling Functions Using Generator h

In this example, we consider to construct a group of sampling functions based on the generating function h in (1). The derived sampling functions are biorthogonal and satisfy all properties in section 2. We define

$$h(t)=\left\{\begin{array}{c} c_n(\varepsilon^2-t^2)^n,\ |t|\le\varepsilon,\\ 0, \qquad \text{else},\end{array}\right.$$

in which $c_n^{-1}=\int_{-\varepsilon}^{\varepsilon}(\varepsilon^2-x^2)^n dx$ is normalizing constants such that $\int_{-\infty}^{\infty}h(t)=1$. Table 1 shows values c_n for selected and n. ε is a parameter, which controls the ability of maximum flat of the sampling functions.

We have,

$$\left.\frac{d^kh}{dt^k}\right|_{x=\pm\varepsilon}=0,\ k<n,$$

$$\begin{aligned}&\frac{d^nh}{dx^n}\\ =\quad & c_n\sum_{k=0}^{n}\frac{(-1)^k(n!)^3}{[k!(n-k)!]^2}(\varepsilon-x)^k(\varepsilon+x)^{n-k},\end{aligned}$$

$$\left.\frac{d^nh}{dx^n}\right|_{x=\pm\varepsilon}=c_n n!2^n(\pm\varepsilon)^n.$$

Consequently, the associated sampling function $\widehat{S}(w)=\int_{w-\pi}^{w+\pi}h(x)dx$, satisfies $\widehat{S}\in C^{n-1}$and

$$\left|\frac{d^kS}{dt^k}\right|\le\frac{C_{k,n}}{(1+|t|)^n},k=0,1,...n, \tag{11}$$

where $C_{k,n}$ is a constant independent of t.

Formula (11) shows the fast decreasing properties of the sampling function. In the case of $n=4$, $\varepsilon=\frac{\pi}{3}$, $C_4=\frac{6200145}{256\pi^9}$, the sampling function S_4 is given by its Fourier transform,

$$\widehat{S}_4(w)=\begin{cases}1,\ |w|\le\frac{2\pi}{3},\\ \frac{1}{256\pi^9}(4\pi-3|w|)^5(368\pi^4\\ -2400\pi^3|w|+5940\pi^2w^2\\ -6615|w|^3+2835w^4),\\ \frac{2\pi}{3}\le|w|\le\frac{4\pi}{3},\\ 0, \qquad \text{else}.\end{cases} \tag{12}$$

S_4 satisfies

$$|S_4(t)|\le\frac{C_{0,4}}{(1+|t|)^4},$$

which is decreasing faster than both of the Shannon sampling function and the Raised cosine sampling funciton in the previous section.

In the case of $n=5,\varepsilon=\frac{\pi}{3},C_5=\frac{122762871}{512\pi^{11}}$, the sampling function S_5 is given by its Fourier transform,

$$\widehat{S}_5(w)=\begin{cases}1, \qquad |w|\le\frac{2\pi}{3},\\ \frac{122762871}{262144\pi^{22}}(-4\pi+3|w|)^5(-1328\pi^5\\ +10656\pi^4|w|-34398\pi^3w^2+\\ 55944\pi^2w^3-45927\pi|w|^4\\ +15309|w|^5),\\ \qquad \frac{2\pi}{3}\le|w|\le\frac{4\pi}{3},\\ 0, \qquad \text{else}.\end{cases} \tag{13}$$

Table 1. Coefficients c_n, $\varepsilon=\frac{\pi}{3}$

n	c_n	n	c_n
4	1.23079854	10	1.42368215
5	1.22701967	11	1.49336209
6	1.24207166	12	1.36315457
7	1.27127839	13	1.66020000
8	1.31210613	14	1.75783327
9	1.36315457	15	1.86549666

4. Conclusion

In this article, we begin with reviewing construction and properties of Meyer type wavelets. We then discussed the sampling functions constructed from the generators of scaling functions. Two interesting examples are given, namely Raised-cosine sampling function and the h-sampling functions. These sampling functions are linear

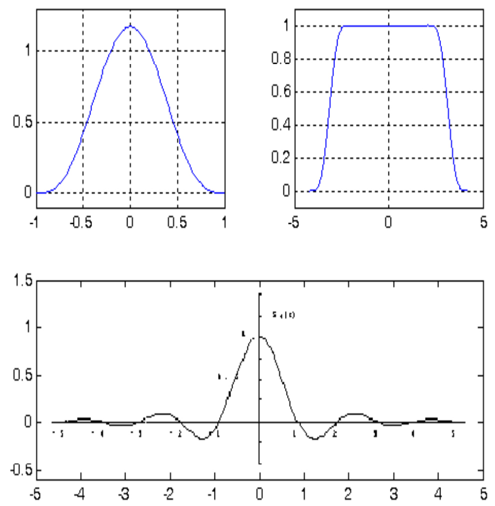

Figure 5: The generator h (left up corner), the associated sampling function $S_4(t)$ in frequency domain (right up corner) and in time domain (bottom), respectively.

phase IIR filters which are easy to implement with adjustable parameters. The h-sampling function has been used to recover discrete bandlimited signals produced by musical instruments. Related results will be reported in future.

REFERENCES

[1] A. Aldroubi and M. Unser. Families of Wavelet Transforms in Connection With Shannon's Sampling Theory and the Gabor transform. Wavelets, 509–528, Wavelet Analysis and Applications, Vol. **2**. Academic Press, Boston, MA, 1992.

[2] A. Bonami, F. Soria and G. Weiss. Band-limited wavelets. Fourier Analysis and Partial Differential Equations (Miraflores de la Sierra, 1992), 21–56, Stud. Adv. Math., CRC, Boca Raton, FL, 1995.

[3] I. Daubechies. *Ten Lectures on Wavelets.* CBMS_NSF Series in Appl. Math., SIAM, Philadelphia, 1992.

[4] M. Frazier, B. Jawerth and G. Weiss. Littlewood-Paley Theory and the Study of Function Spaces. CBMS Regional Conference Series in Mathematics, 79. American Mathematical Society, Providence, RI, 1991.

[5] J. R. Higgins. *Sampling Series in Fourier Analysis and Signal Theory.* Clarendon Press, Oxford, 1996.

[6] Y. Liu and G. G. Walter. Irregular Sampling in Wavelet Subspaces. J. of Fourier Anal Appl., Vol. **2**, 181-189, 1995.

[7] C. E. Shannon. Communication in the Presence of Noise. Proc. IRE, Vol. **37**, 10-21, 1949.

[8] X. Shen. A Quadrature Formula Based On Sampling In Meyer Wavelet Subspaces. J. Comput. Anal. Appl., Vol. **3** (2), 147-163, 2001.

[9] S. Smale and D.-X. Zhou. Shannon Sampling and Function Reconstruction from Point Values. Bull. Amer. Math. Soc. (N.S.), Vol. **41**, 279-305, 2004

[10] G. G. Walter and Shen. Positive Sampling in Wavelet Subspaces. Sampl. Theory Signal Image Process., Vol. **12** (1), 150-165, 2002.

[11] G. G. Walter and X. Shen. *Wavelets and Other Orthogonal Systems. 2nd* ed., CRC Press, Boca Raton, FL, 2001.

[12] G. G. Walter. Approximation With Impulse trains. Result. Math., Vol. **34**, 185-196, 1998.

[13] G. G. Walter and J. Zhang. Orthonormal Wavelets With Simple Closed-Form Expressions. IEEE Trans on Sig. Proc. (1998)

[14] G. G. Walter. A Sampling Theorem for Wavelet Subspaces. IEEE Trans. Inform. Theory, Vol. **38**, 881-884, 1992.

[15] A. Zayed. *Advances in Shannon's Sampling Theory.* CRC Press, Boca Raton, FL, 1993.