MINIMAL ENCLOSING CIRCLE AND TWO AND THREE POINT PARTITION OF A PLANE

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Abstract

Partitions of a plane, based on two or three of its points, are introduced. The study of these partitions is applied to finding the minimal enclosing circle (MEC) for a set S of n planar points. MEC(S), the MEC of n points of S, is defined by either a pair of S points with the largest distance (tight two-tuple) or by a triplet of S points spread on more than half of its circumference (tight three-tuple) with the largest radius. An extension for an existing MEC by an outside point $P \in S$ *is a MEC for point P and the points of the tight tuple for the existing MEC. It has a larger radius than existing MEC. The MEC problem is dual to a problem of finding an optimal partition of S-plane by two or three points of S defined with the largest circular region. A two point partition divides the S-plane in 4 regions, a three point – in 7 regions, one region is a circle in either partition. The MEC algorithm is based on this duality. It begins with a MEC of two arbitrary points of S and corresponding two point partition of S-plane. Next, each point P of S is examined in a separate step of the algorithm. If it is outside of the current MEC, its extension by this point is obtained. The tight tuple for the extension is formed by replacing either none or one or two points of current MEC's tight tuple by point P. Which points of the tuple are to be replaced by point P depends on the region to which P belongs in a plane partition by the points of current tight tuple. The next circle has a larger diameter and it retains at least one set of the defining points of a previous circle, thus limiting a possible loss of its S- points during an extension. A n-step iteration is completed once each point of S is examined. It is repeated until no point of S is found outside of a current MEC during an entire iteration. Observed number of steps in the algorithm has rarely reached 5n and never exceeded 6n in an experiment over several point distributions with n in range from 10 to 28,000,000 . Commonly considered as the fastest,* Cärtner's modification *of Welzl's algorithm [18], [6] has a proved expected performance of O(n). In the experiment MEC algorithm outperformed it in more than 7 times in average. At this point no satisfying theoretical bound, matching this remarkable performance of MEC algorithm, has been found. This incremental algorithm is an on-line algorithm: if set S gets new points during its execution, the current and following iterations continue with an updated set S without a loss of the progress achieved before the update. The algorithm has been already extended to* R^3 *and this will be reported elsewhere.*

This paper is also about the two and three point partitions. They provide the basis for the MEC algorithm.

Key words: *Partition, Minimal Enclosing Circle, Optimization*

Introduction

The MEC problem has a long history. In 1857 Sylvester stated it for a set S of n planar points [15] and in [16] he was discussing efficient solutions for the problem. The MEC problem is an optimization problem and in Operations Research is stated as the minimax facilities location problem:

$$\min_{p_0} \max_{1<=i<=n} ((x_i-x_0)^2+(y_i-y_0)^2)$$

where $p_0(x_0,y_0)$ is the center of the circle to be determined and $S=\{(x_i, y_i)\}$. The solution of the problem is the point p_0 with the minimal greatest distance to any point of S. This criterion is used in emergency facilities, such as police stations and hospitals, to minimize worst-case response time [17], [14].

An improved efficiency for the solution of the MEC problem is achieved with the use of Voronoi partition by Shamos and Hoey [12], [13]. Their algorithm makes use of Voronoi partition of a plane into regions with each region made up by points with the largest distance to a respective one point of S. Once the partition is completed in Ω(n log n) time, it takes O(n log n) time to find the center and radius of the MEC. Megiddo developed a method for finding MEC(S) with optimal linear time based on convex programming and then extended his method to any fixed dimension [7]. In 1991 Emo Welzl developed a randomized method for determining MEC in expected linear time [18]. Algorithms implementing modification of Welzl's method and experimental results are presented in 1997 Cärtner publication [6].

1. Two Points and a Space Partition

Theorem 1

Let A and B be two points in n-dimensional space and O is the middle point of the line segment AB. Let P be an arbitrary point in the space, $\bar{r}_p = \overline{OP}, \bar{R} = \frac{1}{2}\overline{AB}$. Then points A and B induce a unique partition of the space into four regions, the union of which is the space and intersection of each pair of these regions is their boundary. The four regions are defined as follows

(1) $r_p \le R$

(2) $\begin{cases} (\bar{r}_p \cdot \bar{R}) \le -R^2 \\ (r_p \ge R)^* \end{cases}$

(3) $\begin{cases} (\bar{r}_p \cdot \bar{R}) \ge R^2 \\ (r_p \ge R)^* \end{cases}$

(4) $\begin{cases} |(\bar{r}_p \cdot \bar{R})| \le R^2 \\ r_p \ge R \end{cases}$

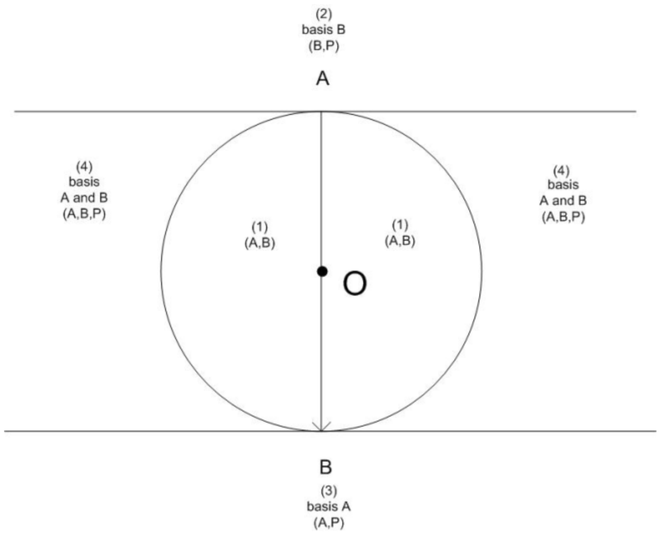
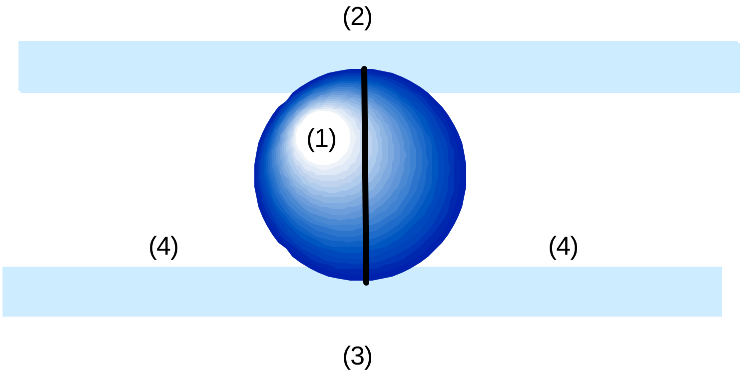


Figure 1. Two-Point Partition of a plane

Proof

The union of points specified by (1) and (4) defines its points with $|(\bar{r}_p \cdot \bar{R})| \le R^2$, while any remaining point of the space is in (2) or in (3). In case of a two-dimensional space, region (1) has the circumference as an intersection with region (4) and a different common single point with region (2) and region (3), while intersection of each with region (4) is a different straight line and intersection of regions (2) and (3) is empty. Vectors $\overline{AB}$ and $\overline{BA}$ define (1) and (4) identically, while regions (2) and (3) for $\overline{AB}$ become regions (3) and (2) for $\overline{BA}$. This means that the partition is uniquely defined by points A and B, while the regions (2) and (3) identifications depend on the tuple choice (A,B) or (B,A) to represent the two-point set {A,B}.



Two-Point Partition of R^3

Definition 1

B, A, A and B are referred to as the basis of the region (2), (3), (4) respectively.

Definition 2

A MEC defined by a tight tuple (A,B,C) or a tight tuple (A,B) is referred to as (A,B,C)-MEC or (A,B)-MEC respectively.

Property of a two-point partition

For a point P outside of (A,B) – MEC, the MEC encircling the three points P,A,B is one of (B,P) –MEC, (A,P)-MEC, (A,B,P)-MEC, depending on P belonging to (2), (3) or (4) respectively. This MEC has a radius larger than OA.

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* It follows from the first inequality in definition of this region

Proof

Figure 1 shows the regions and corresponding MEC tuples. If P belongs to region (2) or (3) the MEC with AP or BP as a diameter contains the third point as a vertex of an obtuse triangle inscribed in it and this diameter is larger than AB. If P belongs to region (4) than (A,B,P)-MEC contains these points and its center lies on a perpendicular to AB at its middle point, making the radius of (A,B,P)-MEC larger than the (A,B)-MEC radius.

2. Three Points and Plane Partition

Theorem 1

Given a circle by its tight three-tuple (A,B,C), there exists a unique partition of its plane in seven regions, the union of which is the plane and an intersection of any pair of the regions is either empty or is their one-dimensional border.

Proof

The partition in Figure 2 shows the circle and the other six regions. This partition is produced by straight lines perpendicular to each side of triangle ABC at their two vertices and the circle. Let $\{P_1, P_2, P_3\} = \{A,B,C\}$ and P be an arbitrary point on the plane. The circle is defined by the three-tuple (A,B,C). Each of the three regions between parallel lines and the circles border is identified by the fact that angles $\angle PP_1P_2$ and $\angle PP_2P_1$ are acute for a point P from the region corresponding to P_1P_2 and extending only from the side of the circle which is further from P_1P_2. Each of the remaining three regions is adjacent to corresponding two regions, considered above, sharing a common linear border with each of them and just one common point with the circle's boundary. A point P belongs to one of these regions if and only if $\angle PP_1P_2$ and $\angle PP_2P_1$ are both obtuse and P_1P_2 and P_1P_3 correspond to the adjacent regions. Each of these regions is defined by a different set of requirements and a non-border point of the plane satisfies one and only one of them.

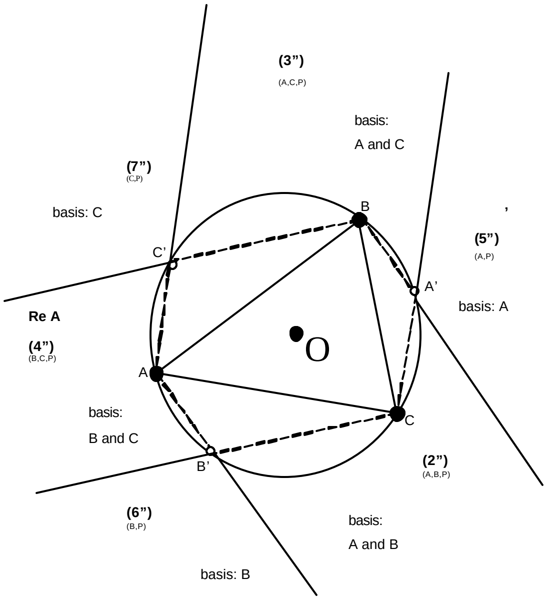
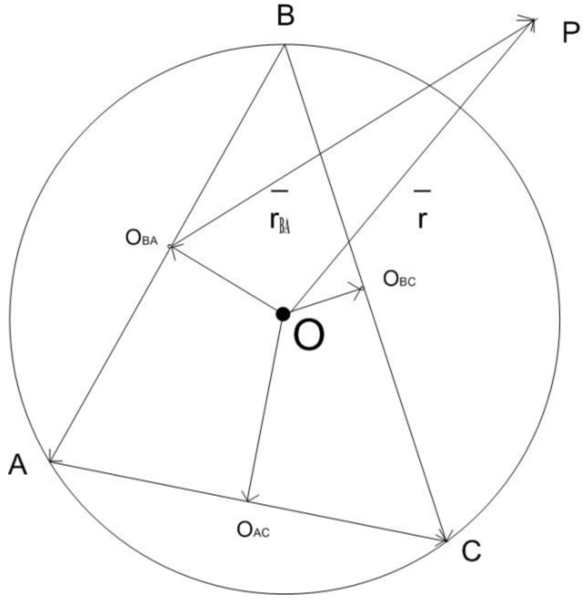


Figure 2. Three- point partition of a plane

Consider



vectors $\overline{AC},\overline{BA},\overline{BC},\ \overline{R}_{AC} = \frac{1}{2}\overline{AC},\ \overline{R}_{BA} = \frac{1}{2}\overline{BA},\ \overline{R}_{BC} = \frac{1}{2}\overline{BC}$, and middle points O_{AC} ,

O_{BA} ,O_{BC} of AC, BA, BC correspondingly. Let $\bar{r} = \overline{OP},\ \bar{r}_{AC} = \overline{O_{AC}P},\ \bar{r}_{BA} = \overline{O_{BA}P},\ \bar{r}_{BC} = \overline{O_{BC}P}$ for an arbitrary point P,

and R be the radius of (A,B,C) – MEC. The regions, specified in Theorem 1, may be also defined as follows:

$(1'')$ r <= R

$$(2'')\ \begin{cases} \left|(\bar{r}_{BA}\cdot\bar{R}_{BA})\right| \le R^2_{BA} \\ (\bar{r}\cdot\overline{O_{BA}O}) \ge 0 \\ r \ge R \end{cases} \qquad (5'')\ \begin{cases} (\bar{r}_{AC}\cdot\bar{R}_{AC}) \ge R^2_{AC} \\ (\bar{r}_{BA}\cdot\bar{R}_{BA}) \le -R^2_{BA} \\ (r \ge R)^{*} \end{cases}$$

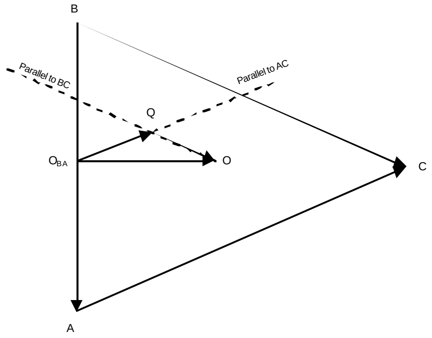
$$(3')\ \begin{cases} \left|(\bar{r}_{AC}\cdot\bar{R}_{AC})\right| \le R^2_{AC} \\ (\bar{r}\cdot\overline{O_{AC}O}) \ge 0 \\ r \ge R \end{cases} \qquad (6'')\ \begin{cases} (\bar{r}_{BC}\cdot\bar{R}_{BC}) \ge R^2_{BC} \\ (\bar{r}_{BA}\cdot\bar{R}_{BA}) \ge R^2_{BA} \\ (r \ge R)^{*} \end{cases}$$

$$(4'')\ \begin{cases} \left|(\bar{r}_{BC}\cdot\bar{R}_{BC})\right| \le R^2_{BC} \\ (\bar{r}\cdot\overline{O_{BC}O}) \ge 0 \\ r \ge R \end{cases} \qquad (7'')\ \begin{cases} (\bar{r}_{AC}\cdot\bar{R}_{AC}) \le -R^2_{AC} \\ (\bar{r}_{BC}\cdot\bar{R}_{BC}) \le -R^2_{BC} \\ (r \ge R)^{*} \end{cases}$$

$(1'')$ specifies the circle.The first inequality in $(2''),(3'),(4'')$ specifies regions (4) of the partitions produced by pairs (B,A), (A,C), (B,C) correspondingly and with the second and third inequality in them the regions are reduced to subregions beyond the circle (O,R) further away from corresponding side of triangle ABC. Regions $(5'')$, $(6'')$, $(7'')$ are intersections of corresponding regions (2) and (3) in the partitions produced by the pairs (A,C), (B,A), (B,C).

Note

Given points A,B,C, parameters defining the regions (1")-(7") may be obtained as described below.



$\overline{O_{BA}O} = \overline{O_{BA}Q} + \overline{QO} = a\cdot\overline{AC} + b\cdot\overline{BC}$.

Vectors $\overline{O_{BA}O}$, $\overline{O_{AC}O}$, $\overline{O_{BC}O}$ may be computed by noticing:

a) Each is perpendicular to an appropriate side of the triangle ABC at its middle point and points to the inside of this triangle since it is acute. Hence, each of these vectors allows to be represented as a linear combination of two vectors with not zero coefficients (positioned on the other two sides of triangle ABC).

b) The magnitude of each (for defining regions $(2'')$, $(3')$, $(4'')$, only directions of these vectors matter) may be computed as $\sqrt{R^2 - \frac{P_iP_j^2}{4}}$, where R is the (A,B,C) – MEC radius, P_iP_j represent an appropriate side of the triangle ABC and $R = \frac{BA\cdot AC\cdot BC}{4S_{\Delta ABC}}$, $S_{\Delta ABC}$ represents the area of triangle ABC and may be computed as $\sqrt{\ell(\ell - BA)(\ell - AC)(\ell - BC)}$ with $\ell = \frac{AB + AC + BC}{2}$. Let $\overline{O_{P_iP_j}O}$ represent one of the three vectors under consideration. According to a) and b) :

(*) $\overline{O_{P_iP_j}O} = \alpha_{ij}\overline{P_iP_k} + \beta_{ij}\overline{P_jP_k}$, where P_k is the remaining vertex of triangle ABC,

(**) $\overline{O_{P_iP_j}O}\cdot\overline{P_iP_j} = 0$,

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* It follows from the other two inequalities in definition of this region

$(***) R^2 - \frac{P_iP_j^2}{4} = O_{P_iP_j}O^2$. Substituting in (**) $\overline{O_{P_iP_j}O}$ by the right side of (*) provides

$\beta_{ij} = -\alpha_{ij}\frac{(\overline{P_iP_k}\cdot\overline{P_iP_j})}{(\overline{P_jP_k}\cdot\overline{P_iP_j})}$ and (***) is producing $R^2 - \frac{P_iP_j^2}{4} = \left(\alpha_{ij}\overline{P_iP_k} - \alpha_{ij}\frac{(\overline{P_iP_k}\cdot\overline{P_iP_j})}{(\overline{P_jP_k}\cdot\overline{P_iP_j})}\overline{P_jP_k}\right)^2$ or

$$(****)\quad \alpha_{ij}^{\,2}\left(P_iP_k^{\,2} - 2\frac{(\overline{P_jP_k}\cdot\overline{P_iP_j})\cdot(\overline{P_iP_k}\cdot\overline{P_jP_k})}{(\overline{P_jP_k}\cdot\overline{P_iP_j})} + \frac{(\overline{P_iP_k}\cdot\overline{P_iP_j})^2}{(\overline{P_jP_k}\cdot\overline{P_iP_j})^2}\right) = R^2 - \frac{P_iP_j^{\,2}}{4}$$

Let T_{ij} be the value for $\alpha_{ij}^{\,2}$ defined by the last equation: $T_{ij} \equiv \alpha_{ij}^{\,2}$.

Notice that the coefficient of $\alpha_{ij}^{\,2}$ in (****) is positive, since it is received as a square of a real value not equal to zero. The selection of the sign of $\pm\sqrt{T_{ij}}$ in defining α_{ij} has to be done according to the directions of the vectors on the sides of triangle ABC. For $\overline{P_iP_j} \equiv \overline{BA}$ the $\alpha_{ij} > 0$, since $\overline{O_{AB}O}$ has an acute angle with $\overline{AC}$; for $\overline{P_iP_j} \equiv \overline{AC}$ the $\alpha_{ij} < 0$, since $\overline{O_{AC}O}$ has an obtuse angle with $\overline{BA}$,and for $\overline{P_iP_j} \equiv \overline{BC}$ the $\alpha_{ij} > 0$ since $\overline{O_{BC}O}$ has an acute angle with $\overline{BA}$.

The β_{ij} is, then, computed by applying the formula $\beta_{ij} = -\alpha_{ij}\cdot\frac{(\overline{P_iP_k}\cdot\overline{P_iP_j})}{(\overline{P_jP_k}\cdot\overline{P_iP_j})}$. To compute each of the vectors $\overline{O_{BA}O}$, $\overline{O_{AC}O}$, $\overline{O_{BC}O}$, one assumes $\overline{P_iP_j} = \overline{BA}$ or $\overline{P_iP_j} = \overline{BC}$ or $\overline{P_iP_j} = \overline{BC}$ or $\overline{P_iP_j} = \overline{AC}$ accordingly, defines T_{ij}, α_{ij}, β_{ij}.

For the purpose of specifying regions ($2''$) - ($7''$) the equations (*) and (**) are sufficient. Together they provide the direction from the middle point of a side of triangle ABC to the (A,B,C) – MEC center. Taking the appropriate sign for α_{ij} and arbitrary its magnitude, the β_{ij} is computed and resulting expression represent a vector of the same direction as $\overline{O_{p_ip_j}O}$. However, assuming the radius-vectors for points of S are given with respect to a fixed point on the plane, in order to define $\bar{r} = \overline{OP}$ at least one of the vectors $\overline{O_{BA}O}$, $\overline{O_{AC}O}$, $\overline{O_{BC}O}$ may be computed – only then the regions ($1''$) – ($7''$) are defined. To define regions ($1''$)-($7''$) in terms of vectors $\bar{r} = \overline{OP}, \overline{BA}, \overline{AC}, \overline{BC}$ only, the second equation in ($2''$), ($3'$), ($4''$) may be replaced by $\left(\bar{r}\cdot\overline{AC}\right) \ge 0$, $\left(\bar{r}\cdot\overline{BC}\right) \le 0$ $\left(\bar{r}\cdot\overline{BA}\right) \ge 0$, and the third equation in (5"), ($6'$), ($7''$) by $(\bar{r}\cdot\overline{BA}) \ge 0, (\bar{r}\cdot\overline{BC}) \ge 0, (\bar{r}\cdot\overline{AC}) \le 0$ respectively. The value of R depends on the values of BA, AC, BC only.

Properties of a Three-Point Partition

Definition

Given a (A,B,C)-MEC, A and B, A and C, B and C, A, B, C are referred to as the basis of the respective region $(2''), (3''), (4''), (5''), (6''), (7'')$ (see Figure 2).

Property 1

A point P of a region with a P_i and P_j basis i,j = 1,2,3, i<j, in a three-tuple partition, produced by a tight tuple (P_1,P_2,P_3), generates a tight three-tuple (P,P_i,P_j) and the (P,P_i,P_j)- MEC encircles all four points (P,P_1,P_2,P_3), (making the radius of the MEC larger than the radius of the (P_1,P_2,P_3)-MEC).

Proof

It follows from the definition of such regions that triangle PP_iP_j is acute. The fact, that $\angle P_iPP_j$ is less than $\angle P_iP_kP_j$ (since P is outside of the (P_i,P_j,P_k) circle) and the center of the new circle is on the same perpendicular to P_iP_j as the center of (P_1,P_2,P_3)-circle, means that to have P_i and P_j on its boundary as well as P it has to shift further from P_iP_j on this perpendicular- resulting in larger radius of the new circle and decreased distance between P_k and the center of the ($P_iP_kP_j$)-MEC.

Property 2

Let P be a point in the region with a single point basis P_i, i=1,2,3, then (P,P_i)-MEC (or (P,P_j)-MEC) correspondingly contains all 4 points P,P_1,P_2,P_3 with a larger than the (P_1,P_2,P_3)-MEC radius.

Proof

In this region the triangle PP_jP_i and triangle PP_kP_i are obtuse and the obtuse angles are at vertices P_j,P_k, {i,j,k} = {1,2,3}. The diameter P_iP_i' of the (P_1,P_2,P_3)-MEC is inside of (P,P_i)-MEC since the later contains the triangle $P_i'P_jP_k$ and point P_i.

3. Extensions

Definition

A MEC is referred to as a P-point extension of Q-MEC, where Q is a tight tuple ,if it is a MEC for points of Q and point P.

Note. An extension operation is essentially an operation on a tight tuple resulting in another tight tuple.The operation is defined by a special transformation of a circle and is determined by an outside point P and two or three points representing the circle. If instead of Q ,a tuple U is chosen to represent the circle Q-MEC, it may result in not the same circle -extension of U-MEC (although U-MEC and Q-MEC coincide geometrically, their extensions may not).

Extension Theorem

Given a Q- MEC, there is one and only one its extension by an outside point P. The radius of the extension is greater than the radius of the Q-MEC.

Proof

If the given MEC is represented by a two point tuple (A,B) then the point P belongs to one of of the three regions of the (A,B)-MEC partition and the tight tuple for the extended MEC is either (A,P) or (B, P) or (A, B, P) depending from which region the point P is drawn. If the given MEC is represented by a three point tuple (A, B, C) then the point P belongs to one of the six regions of this three point partition and the extended MEC is represented by one of six tuples, depending on the region where point P is, (A, B, P), (A, C, P), (B, C, P), (A, P), (B, P), (C,P). The extension has a larger radius, it contains all the points of the MEC tuple and point P outside of it as proved earlier.

The next theorem points to a close relation between a region on a plane and the unique way producing an extension by any point of this region. It allows for an equivalent definition of regions as subsets of plane points, each requiring the same basis for producing an extension. It immediately follows from partition and extension properties above.

Equivalency Theorem

For points of a plane to belong to the same region of an induced by a Q- tuple partition , it is necessary and sufficient that any extension of Q-MEC by one of these points may be defined by a tuple differing in this point only (made up from this point and the same basis).

Duality Theorem

Minimal enclosing circle problem is dual to a problem of finding two (three) points in S that induce a partition of the S-plane with the largest region 1 (region $1''$).

Proof

If MEC(S) is found and (A,B) or (A,B,C) is its tight tuple of points of S then a (A,B)-partition or (A,B,C)-partition of the plane is defined with the largest possible radius for the circle in region 1 or region $1''$. If a partition of the plane with the largest region 1 or region $1''$ is defined by 2 or 3 points of S the points are easily identified (see Figure 1, Figure 2) and, since they constitute a tight tuple for the largest circle in a two point or a three point partition of the plane produced by points of S, the circle does not have a point of S outside it.

4. MEC Algorithm

Initialization:

Let T be the array containing all n planar points of S and $P^{(1)}$, $P^{(2)}$ be the two arbitrary points of it. Assume NP=2, MEC=($P^{(1)}$,$P^{(2)}$)-MEC

Iterative part:

```
do
    AMEC=MEC;
    for (i=0; i<n; i++)
    {
        if (NP==2)
            compute which of the 4 regions[†] holds point T[i] and if it is not the region(1), compute the
            extension of the MEC by point T[i] and update NP to 3 if the region is (4);
        else
            compute which of the 7 regions[††] holds point T[i] and if it is not the region ($1''$), compute the
            extension of the MEC by point T[i] and update NP to 2 if the above region is either ($5''$)or ($6''$)
            or ($7''$);
        end if
        MEC=extension
    }
while (MEC is not coinciding with AMEC).
```

[†] See Figure 1

[††] See Figure 2

Convergence of MEC Algorithm

Theorem

Generated by the algorithm sequence of extensions $E = E_1, E_2, E_3, \ldots$ converges to MEC(S).

Proof

An extension E_{t+1} has a larger radius than extension E_t, $t=1,2,\ldots$, and is defined by a tight tuple and consequently the same tuple does not occur twice in E. The algorithm generates extensions until no point of S is outside of a current extension. The fixed amount of tuples ensures that this process is finite and delivers MEC(S).

5. Experimental Results

Points for 48 sets for a distribution were generated by a random number generator with n in range between 10 and 28,000,000. Several different point distributions were used and the observed number of steps has rarely reached 5n and never exceeded 6n. Gärtner's modification of Welzl's algorithmwas used as a benchmark in this experiment. In average MEC algorithmsolved the same problem in more than 7 times faster. For example ,it took 3.45 seconds for the MEC algorithm as opposed to 28.64 seconds for Gärtner's modification to compute the same optimal solution for a 28,000,000 point instance . A 1999 Gärtner's C++ program, available on his website ,and my C++ program were used in the experiment, delivering the same optimal solutions for each instance.

6. Conclusion

A new method for solving the MEC problem and a new method for a plane partitioning are introduced. The latter is incorporated in the MEC algorithm. Experimental results show more than 7 times faster performance of MEC algorithm when compared to the commonly considered fastest method .

Bibliography

[1] Clarkson K., P.W. Shor, Application of random sampling in computational geometry, Disc. Comp. Geom., 4, pp. 387-421, 1989.

[2] Elzinga D.J., D.W. Hearn, Geometrical Solutions for some minimax location problems, Transportation Science 6, pp. 379-394, 1972.

[3] Elzinga D.J., D.W. Hearn, The minimum covering sphere problem, Mgmt Sci. 19 (1), pp. 96-104, 1972.

[4] Francis R.L., J.A. White, Facility Layout and Location, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[5] Francis R.L, Some aspects of a minimax location problem, Operat. Res, 15 (1967), pp. 1163-1168.

[6] Gärtner B., Geometric Optimization. Lecture notes for the Equinoctial School on Geometric Computing, ETH Zürich, 1997.

[7] Megiddo N., Linear-time algorithms for linear programming in R^3 and related problems, SIAM *Journal on Computing* 12 (1983), pp. 759-776.

[8] Nair K.P.K, R. Chandrasekaran, Optimal location of a single service center of certain types, Naval Research Logist.

Quart, 18 (191), pp. 503-510.

[9] O'Rourke J., Computational Geometry in C, Cambridge University Press, 1994.

[10] Preparata F.P., M.A. Shamos, Computational Geometry: An Introduction, Springer Verlag 1988.

[11] Rademacher H., O. Toeplitz, The Enjoyment of Mathematics, Princeton Univ. Press, Princeton, NJ, 1957.

[12] Shamos M.I., Computational Geometry, Ph.D thesis, Dept of Computer Sci., Yale Univ, 1978.

[13] Shamos, M.I. and D. Hoey, Closest-Point Problems, in Proc. 16th Annual IEEE Symposium on Foundations of Computer Science, 1975, IEEE Computer Society Press, Los Angelos, 1975, pp.151-162.

[14] Smallwood, R.D., Minimax Detection Station Placement, Oper. Res., 13 (1965), pp. 636-646.

[15] Sylvester J. J., *A question in the geometry of situation*, Quart. J. Math., 1 (1857), p. 79.

[16] Sylvester J.J., *On Poncelet's approximate valuation of Surd forms*, Philosophical Mag., XX, 4th Series (1860), pp. 203-222.

[17] Toreqas C., R. Swain, C. Revelle, L. Bergman, The location of emergency service facilities, Operations Research, 19, pp. 1363-1373 (1971).

[18] Welzl Emo, Smallest enclosing disks (balls and ellipsoids). In H. Maurere, editor, *New Results and New Trends in Computer Science*, volume 555 of *Lecture Notes in Comput.Sci.*, pp. 359-370. Springer-Verlag, 1991.