Optimal Automatic Control Solution to Nonanticipating Operator Dynamical Systems

Reza Ahangar* Ebrahim Salehi**

* Department of Mathematics, Kansas Wesleyan University, 100 E. Claflin Ave, Salina, KS 67401

email: rahangar@kwu.edu, Phone: (785)-827-5541, ext 2420, Fax: (785)-493-8411

** Department of Mathematics, University of Nevada Las Vegas, Las Vegas, NV 8 9154.

email:esalehi@ccmail.nevada.edu, Phone:(702)-895-3567, Fax:(702)-895-4343

Abstract: A review of a unified approach to nonlinear operator dynamical systems is presented. The solution to a system of nonanticipating operator differential equations applied to the automatic control system is studied. Conditions for existence and uniqueness of the solution for optimal control systems are provided. The global solution to the optimal automatic control of nonlinear nonanticipating dynamical systems is investigated.

Key Words: nonanticipating, Lipschitzian, and induced operators, automatic optimal controls.

The 2006 World Congress in Computer Science, Computer Engineering, Applied Computing
Conference on Scientific Computing (CSC'06)

1.0 Introduction and motivation:

By **Generalized Dynamical Systems**, we mean a nonlinear operator differential equation

$$y\prime(t) = f(t, y(t), T(y)(t)) \qquad for \text{ all } \quad t > t_0 \tag{1.1}$$

and $y(t) = \phi(t)$ for all $t \leq t_0$. This dynamical system could be interpreted and applied to a certain operator satisfying the conditions for the solution.

We will study the generalized dynamical systems with **nonanticipating and Lipschitzian operator** applied to optimal automatic control problems.

Basic Definitions: In what follows Y, Z, and U will be Banach Spaces, and $I = [O, a]$, $S = \{t : t < O\}$, and $J = I \cup S$ for an arbitrary but fixed real number a. Denote by $M(K, Y)$ the Banach space of all essentially bounded Bochner measurable functions from an interval K into a Banach space Y with respect to classical Lebesgue measure. The norm in this space will be the essential supremum norm $\| . \|_\infty$.

Define $L(K, Y)$ to be the Banach space of all Lipschitzian functions $y : K \to Y$ **strongly differentiable almost everywhere on** K. For a fixed initial function $\phi \in L(S, Y)$ we define the **Initial Domain**, denoted by $D(\phi, Y)$, to be the subset of $L(S, Y)$ consisting of all functions y such that $y(t) = \phi(t)$ for all $t \in S$. That is, $D(\phi, Y) = \{y \in L(J, Y) : y|_s = \phi\}$.

These initial domains can be considered a subset of the space $M(I, Y)$. Lipschitzian Space (or simply **Lip-Space**), denoted by $Lip(K, Y; Z)$, is the set of all functions $f : K \times Y \rightarrow Z$ such that $f(t, y)$ is uniformly bounded, Lipschitzian in y, and measurable in t.

The infimum of all Lipschitzian constants L will be denoted by $\|f\|$. An operator P from a subset D of Y into Z is said to be Lipschitzian if there exists a constant b such that $|P(y_1) - P(y_2)| \leq b|y_1 - y_2|$ for every $y_1, y_2 \in Y$.

On the space M(I,Y), we shall introduce a family of norms, called **k-norms** by the formula $\| y \|_k = ess.\sup\{e^{-kt}|y(t)| : t \in I\} < \infty$, for any fixed real number k. From this definition follows the inequality $|y(t)| \leq \| y \|_k \, e^{kt}$ for almost all t in I. Notice that for every k, the k norms $\| \ \|_k$and $\| \ \|_0$are equivalent. A Lipschitzian operator P from a subset D of M(J,Y) into the space M(I,U) is called **an operator of exponential type,** if for some constants b and k_0,

$$\| P(y) - P(z) \|_k \leq b \| y - z) \|_k \tag{1.2}$$

for all y and z in the domain D and all $k \geq k_0$. For $f \in Lip\,(I, Y; Z)$ the operator $F : M(I, Y) \rightarrow M(I, Z)$ if and only if $F(y)(t) = f(t, y(t))$ is called the induced operator **generated** by f.

Properties of the nonlinear operator F induced by the function f have been studied in Bogdan (1982). In particular, we know that, for $f \in Lip\,(I, Y; Z)$ and $y \in M(I, Y)$, the function $g : I \rightarrow Z$ defined by $g(t) = f(t, y(t))$ belongs to the space of measurable functions $M(I, Z)$.

An operator $T : D(\phi, Y) \rightarrow M(I, Z)$ is called **nonanticipating** if, for every two functions $y, z \in D(\phi, Y)$ and every $s \in I$, the fact that $y(t) = z(t)$ for almost all $t < s$ implies that $T(y)(t) = T(z)(t)$ for almost all $t < s$.

When an operator T is nonanticipating, the future values of the input will have no effect on the present state. One can prove that the delay and the integral operators are Nonanticipating and Lipschitzian. Similarly, the composition and Cartesian product of nonanticipating and Lipschitzian operators are Nonanticipating and Lipschitzian. Furthermore, the operator F induced by the function f is a well defined, nonanticipating, and Lipschitzian operator.

2.0 Solution to the Nonlinear Operator Differential Equation:

Given a nonanticipating and Lipschitzian operator $G : D(\phi, Y) \rightarrow M(I, Y)$, there exists a unique solution $y \in D(\phi, Y)$ such that

$$y'(t) = G(y)(t) \qquad \text{for almost all t} \in I. \tag{2.1}$$

The proof of existence and uniqueness of the solution to the nonlinear operator differential equation is presented in Bogdan (1982), Ahangar 1989, and Ahangar-Salehi 2002.

Theorem 2.1. (Solution to the Operator Control Differential Equations): Let $f \in Lip(I, Y \times Z \times U; Y)$. If $T : D(\phi, Y) \rightarrow M(I, Z)$ is a nonanticipating and Lipschitzian operator then, for any control function $u \in M(I, U)$, there is a unique solution $y \in D(\phi, Y)$ satisfying the state equation of the generalized dynamical system

$$y'(t) = f(t, y(t), T(y)(t), u(t)) \tag{2.2}$$

for almost all $t \in I$.

Sketch of the Proof: For a given admissible control $u \in M(I, U)$ define operators $E : M(I, Y) \to M(I, Y) \Leftrightarrow E(y) = y$ and $U_u : D(\phi, Y) \to D(\phi, Y) \Leftrightarrow U_u(y) = u$.

The Cartesian product $g = E \times T \times U_u \Leftrightarrow g(y) = (E(y), T(y), U_u(y))$ is nonanticipating and Lipschitzian. Now we define operator G by $G(y)(t) = f(t, g(y)(t)) \Leftrightarrow G = F \circ g$ for all $y \in D(\phi, Y)$ and all $t \in I$. Notice that the nonlinear operator F is the operator induced by the function $f \in Lip(I, Y \times Z \times U; Y)$. Since F is nonanticipating and Lipschitzian, so is the composition operator G. Therefore, the differential equation (2.2) is equivalent to the system (2.1) for almost all $t \in I$ and has a unique solution in $D(\phi, Y)$ satisfying the differential equation (2.2)► .

We will now investigate the solution to a dynamical system

$$y\prime(t) = f(t, y(t), w(t)) \tag{2.3}$$

for almost all t in I, with an automatic control relation defined by

$$w = W(y) \Leftrightarrow w(t) = g(t, y(t), T(y)(t)). \tag{2.4}$$

This will lead us to the solution of the nonanticipating operator differential equation in the following theorem.

Corollary 2.1: Let $f \in Lip(I, Y \times Z \times U; Y), g \in Lip(I, Y \times Z; Y)$, and T be a nonanticipating and Lipschitzian operator. Then, for every initial function ϕ, there exists a unique solution y in the initial domain $D(\phi, Y)$ satisfying the nonlinear dynamical system (2.3) and the **automatic control relation** (2.4)

Automatic Controls: One application of the solution of automatic controls in Proposition 2.1 is the dynamical system involving control u.

In this case, the solution to the automatic controls depends on a parameter u(t). Suppose operator T is nonanticipating and Lipschitzian from the initial domain $D(\phi, Y)$ into the space $M(I, Z)$. We can prove that, for any given admissible control function u from the space of $M(I, U)$, there exists a unique solution to the dynamical system

$$y\prime(t) = f(t, y(t), w(t), u(t)), \qquad for \text{ all } \; t \in I, \tag{2.5}$$

and $y(t) = \phi(t)$ for almost all $t < 0$, with a given automatic control relation

$$\mathbf{w(t) = g(t, y(t), T(y)(t))}. \tag{2.6}$$

Theorem 2.2: Given the admissible control function $u \in M(I, U)$. Assume function g is defined in the space $Lip(I, Y \times Z; Z)$ *and* $f \in Lip(I, Y \times Z \times U; Y)$. There exists a unique solution $y \in D(\phi, Y)$ to the system (2.5) which satisfies the relation (2.6).

Sketch of the Proof: By introducing an operator W as defined in relation

(2.4) and substituting w in the relation (2.5) we produce an equivalent composition operator. The operator $W = g \circ (E \times T)$ is a composition and product of the nonanticipating and Lipschitzian operators. Let us introduce an operator $G_u = f \circ (E \times W)$ induced by the function f. Thus, the equivalent operator dynamical system

$$y'(t) = G_u(y)(t) \tag{2.7}$$

has a unique solution y in the initial domain $D(\phi, Y)$ ▶.
The following is a direct result of Theorem 2.2.
Corollary 2.2: Given an admissible function u, the following integral equation

$$y(t) = \phi(0) + \int_0^t f(s, y(s), w(s), u(s))ds \tag{2.8}$$

is equivalent to the system (2.5) and has a unique solution $y \in D(\phi, Y)$.
As a result, for every admissible control function u, the following automatic control dynamical system

$$y\prime(t) = f(t, y(t), T(y)(t), w(t), u(t)), \qquad t \in I, \tag{2.9}$$

$$w(t) = g(t, y(t), T(y)(t)) \tag{2.10}$$

$y(t) = \phi(t), t < 0$ has a unique solution y(t) in the initial domain $D(\phi, Y)$. Note that in the system (2.9) - (2.10) the function w(t) is an automatic control, but u(t) is any admissible control function.

3.0 Optimal Automatic Controls

In the previous section we investigated the solution to the dynamical system (2.9)-(2.10). Since for every admissible control $u \in M(I, U)$ there exists a unique solution y in the initial domain $D(\phi, Y)$, the mapping

$$\Theta : M(I, U) \to D(\phi, Y) \qquad \Leftrightarrow \quad \Theta(u) = y \tag{3.1}$$

is well defined. In this section we are looking for a control function u satisfying the automatic control system (2.9)-(2.10) and optimizing the functional

$$p(u) = \int_I f_0(t, y(t), T(y)(t), u(t))dt. \tag{3.2}$$

Continuity of the Operator Θ : In the following proposition we will prove the continuity of the Mapping Θ from the control function u into the space of Lipschitzian Trajectory.
Proposition 3.1: Suppose that the operator T from the initial domain $D(\phi, Y)$ into the space M(I,Z) is nonanticipating and Lipschitzian. Assume that the function f belongs to the space $Lip(I, Y \times Z \times U; Y)$. Then the operator Θ

defined in (3.1) is mapping a function u from $M(I, U)$ into the space $D(\phi, Y)$ such that $y = \Theta(u)$ is continuous from M(I,U) with the L-topology into the space $C(I, Y)$ of a continuous function with the usual supremum norm topology.

Sketch of the Proof: According to Theorem 2.2, given control functions u_1 and u_2 in the space $M(I, U)$, there exists unique corresponding solutions y_1 and y_2 in $D(\phi, Y)$ to the system (2.9)-(2.10). Using the integral equation (2.9), k-norm, exponential type operators, and the fact that f belongs to the Lip-space, we can show that

$$\| \Theta(u_1) - \Theta(u_2) \|_k = \| y_1 - y_2 \|_k < b \| u_1 - u_2 \|_k \tag{3.3}$$

for all u_1 and u_2 in M(I,U) and $b = \frac{k\|f\|}{k - \|f\|(1+c)}$. For further details about the estimate see Bogdan (1982) and Ahanagr (1989). This proves that the operator Θ is Lipschitzian with respect to the topology generated by the L-seminorm on the space M(I,U).

Continuity of the Performance Index: For a function $f_0 \in Lip(I, Y \times Z \times U; R)$ we define the **performance index** p generated by f_0 as

$$p(u) = \int_I f_0(t, y(t), T(y)(t), u(t))dt \tag{3.4}$$

for every $u \in M(I, U)$. In the following proposition we will prove that the performance index $p : M(I, U) \rightarrow R$ is a continuous functional from the space $M(I, Y)$ with L-topology into the set of real numbers.

Proposition 3.2: Assume $f_0 \in Lip(I, Y \times Z \times U; R)$ and $T : D(\phi, Y) \rightarrow M(I, Z)$ is nonanticipating and Lipschitzian, then the functional p defined by (3.4) is continuous.

Sketch of the Proof: Suppose we are given two control functions u_1 and u_2 in the space of $M(I, U)$. Let $y_1 = \Theta(u_1)$ and $y_2 = \Theta(u_2)$ denote the corresponding solutions to the differential equations (2.9)-(2.10) for all $t \in I = [0, a]$. Using the performance index (3.2) and (3.3), one can estimate

$$| p(u_1) - p(u_2) | \leq m. \| u_1 - u_2 \| \tag{3.5}$$

where $m = (\frac{ak\|f_0\|(1+\|T\|_k).k\|f\|}{k-(1+c)\|f\|} + 1) \| f_0 \|$. For details see Bogdan (1981) and Ahangar (1986). Hence the functional p is Lipschitzian in the space M(I,U) with the L-topology (topology of the Bochner summable functions). It is continuous on the set M(I,U) with respect to L-topology ▶.

Optimal Solution to Nonlinear Automatic Control Systems: According to the result of the previous section, for a given control function u in $M(I, U)$ there exists a unique solution $y \in D(\phi, Y)$ satisfying the automatic control systems (2.9)-(2.10). Introduce a functional

$$p(u) = \int_I f_0(t, y(t), T(y)(t), u(t))dt \tag{3.6}$$

as a **performance index** for all $u \in M(I, U)$ and $y = \Theta(u)$.
In the following theorem we shall prove that there exists an optimal control u satisfying the automatic control system (3.6) and optimizing the functional p on a suitable set of admissible controls.

Theorem 3.1 : Let $f \in Lip(I, Y \times Z \times U \times U; Y)$, $g \in Lip(I, Y \times Z, Y)$, and $f_0 \in Lip(I, Y \times Z \times U; R)$, and T be as before. If Q is a compact subset of M(I,U) in the L-topology (topology of summable functions), then there exists an optimal control $u \in Q$ satisfying the dynamical system (3.6) and optimizing the functional p defined in (3.6).

Proof: Consider a nonlinear automatic control defined by the operator W

$$w = W(y) \Leftrightarrow w(t) = g(t, y(t), T(y)(t)) \tag{3.7}$$

for every $t \in I$. Substitute the operator W(y) from (2.10) into the relation (2.9). Thus $y\prime(t) = f(t, y(t), T(y)(t), W(y)(t), u(t))$.
The operator generated by g is equivalent to the Cartesian product of operators $g \circ (E \times T)$ and f is equivalent to $E \times T \times W \times U_u$. Define an operator $G_u \equiv f \circ (E \times T \times (g \circ (E \times T) \times U_u))$.
The automatic control system (2.9) and (2.10) will be equivalent to $y'(t) = G_u(y)(t)$ for $t \in I$ and $y(t) = \phi(t)$ for almost all $t < 0$. For any given admissible control function $u(t)$, there exists a unique trajectory y in the initial domain $D(\phi, Y)$ satisfying the automatic dynamical system (2.9)- (2.10).
The existence and uniqueness of the solution y in the initial domain $D(\phi, Y)$, satisfying the dynamical system (2.9) and the automatic control relation (2.10), leads us to introduce an operator that depends

on a control parameter u in the system (2.10). The dynamical system (2.10) will be equivalent to

$$y(t) = \phi(0) + \int_0^t f(s, y(s), T(y)(s), z(s), u(s))ds \tag{3.8}$$

for all t in I, where $z = W(y)$. The argument in Proposition 2.2 and also the previous conclusions verify that, for every given admissible control function u from the space M(I,U), there exists a unique function y in the initial domain $D(\phi, Y)$. This result introduces an operator $\Theta : M(I, U) \to D(\phi, Y)$ such that $\Theta(u) = y$. This operator is well defined in the topology generated by the space $L(v, Y)$ of Bochner summable functions.
By Proposition 3.1, the mapping $\Theta : u \mapsto y$ is continuous. Since for every $u \in Q$ the solution to the system (2.9)-(2.10), $y \in D(\phi, Y)$ is well defined. Thus the functional $p : Q \mapsto R$ on the compact domain Q will take its maximum or minimum. Therefore there exists an optimal control solution $u \in Q$ satisfying the automatic control relation ▶.

4.0 Conclusions:

The first result of this paper is the existence and uniqueness of the solution to the automatic control systems (2.9)- (2.10) which arises from the model of spaceship navigation. Furthermore, we assume that the dynamical system (2.10) depends on another control parameter u from the space M(I,U) that needs to be determined in order to satisfy

$$y\prime(t) = f(t, y(t), T(y)(t), w(t), u(t)) \tag{4.1}$$

with automatic controls given by

$$w(t) = g(t, y(t), T(y)(t)) \tag{4.2}$$

for almost all $t \in I$ such that
$y(t) = \phi(t)$, for all $t < 0$ and optimizing the cost function

$$p(u) = \int_I f_0(t, y(t), T(y)(t), u(t))dt \tag{4.3}$$

The regularity conditions for functions f, g, f_0, and the operator W are the following: functions f, g, and f_0 which are essentially bounded Bochner measurable in $t \in I$, and Lipschitzian with respect to other variables. The operator W is nonanticipating and Lipschitzian, from the initial domain $D(\phi, Y)$ into the space of Bochner measurable functions, for every initial function ϕ.

References

[1] Ahangar R., and Salehi, E., "Automatic Controls for Nonlinear Dynamical Systems with Lipschitzian Trajectories", Journal of Mathematical Analysis and Application, JMAA, 268, pp. 400-405, 2002.

[2] Ahangar R., “Nonanticipating Dynamical Model and Optimal Control”, Applied Math Letter, vol. 2, No.1, pp. 15-18, 1989.

[3] Bogdan V.M., “Existence and Uniqueness of Solution to Nonlinear Operator Differential Equations Generalizing Dynamical Systems of Automatic Spaceship Navigation,” Nonlinear Phenomena in Mathematical Sciences, Academic Press, pp. 123-136, 1982.

[4] Driver R. D., Ordinary and Delay Differential Equations”, Springer Verlag, 1977.

[5] Hale J., ”Theory of Functional Differential Equations”, Spring Verlag 1977.

[6] Lakshmikantham, V. and Leela S., ”Nonlinear Differential Equations in Abstract Spaces”, Pergamon Press 1981.