Application of a newly designed super element to modal analysis of hollow cylinders*

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Abstract

Design and application of super elements in efficient prediction of the structural behaviour in a short time has been one of the research interests in the last decade. A new 16-node cylindrical super element is presented in this paper and implemented to modal analysis of hollow cylinders subjected to various boundary conditions. Comparison of the findings with conventional finite element reveals CPU time saving and accuracy of the results. It is seen that this element can predict the vibration behaviour of hollow cylinders in an efficient manner.
Keywords: FEM; Super Element; Hollow cylinder; Cylindrical element; Modal analysis

1. Introduction

The finite element method (FEM) provides a mathematically simplified procedure to simulate and analyse complex structures. However the FEM becomes time consuming as the number of elements meshing the entire structure increases. Decreasing the number of elements and degrees of freedom (DOF) without altering the accuracy of the findings will results in reduction or the required storage capacity and run time consumption. Several methods to reduce the number of DOFs and increase the computational efficiency in both static and dynamic analysis of solid structures have been presented in the past. The spectral finite element method (SFEM) [1] and the transfer matrix method [2] are examples of these efforts. Development and application of super elements in structural analysis of various mechanical systems have been widely extended in the last decade. F. Ju and Y.S. Choo [3] developed a super element for modelling a cable passing through multiple pulleys. J. Jiang and M.D. Olson [4] extended a super element to the nonlinear static and dynamic analysis of orthogonally stiffened cylindrical shells. Application of geometrical super elements in large deformation of elasto-plastic shells is presented by S.A. Lukasiewicz [5]. Using super elements has also been adopted to structural analysis of buildings. H.S. Kim and D.G. Lee [6] used a rectangular super element to analyze shear walls with openings in a building. Many researchers have used super elements in static and dynamic analysis of stiffened shells and plates. Koko and Olson [7] used a rectangular super element, which is a macro-element having analytical as well as the usual finite element shape functions for vibration analysis of isotropic stiffened plates. Koko and Olson [7-9], Jiang and Olson [10, 11], Vaziri *et al.* [12] have used this element for the analysis of plates and shells.M.T. Ahmadian and M.S. Zangeneh [13,14] have implemented this element in the dynamic analysis of laminated composite plates. Laminated cylinders are basic members in many structures and mechanical systems. They are used as the main part of motors and rotary system hence accurate prediction of their behaviour is of considerable importance. In this paper a new cylindrical super element is introduced. Application of this element has reduced the required computational time while preserving the accuracy of the findings.

2. Element Definition

Consider a cylindrical element with length, $2L$, inner radius r_1 and outer radius r_2, which is depicted in Fig.1. As it is shown it is assumed 8 number of nodes on each side of the element which constitute a total number of 16.

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2.1. *Intrinsic Coordinates*

Since the element is 3 dimensional, 3 independent intrinsic coordinates are required to completely describe the position in the element. These coordinates are defined as follows:

$$\xi = \frac{z}{L}$$

$$\eta = \frac{2 \times r - b}{a} \qquad (1.)$$

$$\gamma = \frac{\alpha}{\pi} - 1$$

where

$$a = r_2 - r_1$$

$$b = r_2 + r_1 \qquad (2.)$$

and $2L$ is total element length.

Considering the appropriate limits for global coordinates;

$$-L \le z \le L$$

$$r_1 \le r \le r_2 \qquad (3.)$$

$$0 \le \alpha \le 2\pi$$

implies;

$$-1 \le \xi, \eta, \gamma \le 1 \qquad (4.)$$

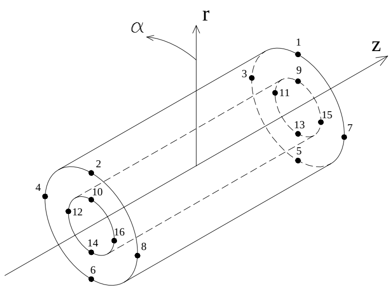

Fig. 1. Super element configuration and coordinate system

2.2. *Shape functions*

A set of 16 shape functions are used to approximate the field variable, i.e. the displacement field in the element. The shape functions are continuous differentiable functions that must satisfy the following condition;

$$N_i(\xi,\eta,\gamma)_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad i, j = 1, 2, \ldots, 16 \qquad (5.)$$

where $(\xi,\eta,\gamma)_j$ indicates the local coordinate for node j and

$$-1 \le \xi, \eta, \gamma \le 1 \qquad (6.)$$

Many sets of shape functions can be used in Eq. 5 which are not stated here for the sake of brevity.

3. Stress-strain relations

The displacement vector in an interior point of the element is

$$\mathbf{u} = \begin{bmatrix} u_r & u_\alpha & u_z \end{bmatrix}^T \qquad (7.)$$

which is obtained using the shape functions and the nodal displacement vector, $\mathbf{q}$ according to

$$\mathbf{u} = \mathbf{N}\,\mathbf{q} \qquad (8.)$$

where

$$\mathbf{q} = \begin{bmatrix} u_{1r} & u_{1\alpha} & u_{1z} & \ldots & u_{16r} & u_{16\alpha} & u_{16z} \end{bmatrix}^T \qquad (9.)$$

and

$$\mathbf{N}=\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & N_{16} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_{16} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & 0 & 0 & N_{16} \end{bmatrix} \qquad (10.)$$

and each of the N_i are the shape functions.

In cylindrical coordinate system the strain vector is defined as;

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_r & \varepsilon_\alpha & \varepsilon_z & \gamma_{r\alpha} & \gamma_{zr} & \gamma_{\alpha z}\end{bmatrix}^T \qquad (11.)$$

where each of the ε and γ correspond to normal and shear stresses respectively.

The strain-displacement relations in cylindrical coordinate are; [15]

$$\begin{aligned}
\varepsilon_r &= \frac{\partial u_r}{\partial r} \\
\varepsilon_\alpha &= \frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\alpha}{\partial \alpha} \\
\varepsilon_z &= \frac{\partial u_z}{\partial z} \\
\gamma_{r\alpha} &= \frac{1}{r}\frac{\partial u_r}{\partial \alpha}+\frac{\partial u_\alpha}{\partial r}-\frac{1}{r}u_\alpha \\
\gamma_{zr} &= \frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r} \\
\gamma_{\alpha z} &= \frac{\partial u_\alpha}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \alpha}
\end{aligned} \qquad (12.)$$

which can be stated in matrix form as;

$$\boldsymbol{\varepsilon}=\mathbf{L}\,\mathbf{u} \qquad (13.)$$

where $\mathbf{L}$ is the operator matrix;

$$\mathbf{L}=\begin{bmatrix} \frac{\partial}{\partial r} & \frac{1}{r} & 0 & \frac{1}{r}\frac{\partial}{\partial \alpha} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{1}{r}\frac{\partial}{\partial \alpha} & 0 & \frac{\partial}{\partial r}-\frac{1}{r} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \alpha} \end{bmatrix}^T \qquad (14.)$$

and $\mathbf{u}$ is the displacement vector defined by Eq. 7.

The strain vector may be written as

$$\boldsymbol{\varepsilon}=\mathbf{B}\,\mathbf{q} \qquad (15.)$$

where $\mathbf{B}$ is the Strain-nodal displacement matrix.

Using Eqs. 9, 13 and 15 the $\mathbf{B}$ matrix for the element e is obtained as;

$$\mathbf{B}^e_{6\times 48}=\mathbf{L}_{6\times 3}\times \mathbf{N}^e_{3\times 48} \qquad (16.)$$

The stress vector in cylindrical coordinate system is defined as

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_r & \sigma_\alpha & \sigma_z & \tau_{r\alpha} & \tau_{zr} & \tau_{\alpha z}\end{bmatrix}^T \qquad (17.)$$

which is related to the strain vector according to the following equation

$$\boldsymbol{\sigma}=\mathbf{D}\boldsymbol{\varepsilon} \qquad (18.)$$

in which $\mathbf{D}$ is the symmetric material property matrix given by [15]

$$\mathbf{D}=\frac{E}{1+\nu}\begin{bmatrix} d & e & e & 0 & 0 & 0 \\ e & d & e & 0 & 0 & 0 \\ e & e & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \quad , d=\frac{1-\nu}{1-2\nu}, \; e=\frac{\nu}{1-2\nu} \qquad (19.)$$

where E and ν are modulus of elasticity and Poisson's ratio respectively.

4. Element stiffness matrix

The element stiffness matrix $\mathbf{K}^e$ is defined as; [16]

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^{eT} \mathbf{D} \mathbf{B}^e dv \quad (20.)$$

where the integration is carried over the element volume V^e. The infinitesimal volume dv in the cylindrical coordinate system can be expressed as;

$$dv = r \cdot d\alpha \, dr \, dz \quad (21.)$$

To transform the integral 20 from (r, α, z) space to (ξ, η, γ) space the Jacobian of the transformation is needed. Here the infinitesimal volume in local coordinate system can be expressed as;

$$dv_{local} = \left|\det(\mathbf{J})\right| \cdot dv_{global} \quad (22.)$$

where the Jacobian is;

$$\mathbf{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial \alpha}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial \alpha}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial r}{\partial \gamma} & \frac{\partial \alpha}{\partial \gamma} & \frac{\partial z}{\partial \gamma} \end{bmatrix} \quad (23.)$$

Using Eq. 1, the Jacobian becomes;

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & L \\ \frac{a}{2} & 0 & 0 \\ 0 & \pi & 0 \end{bmatrix} \quad (24.)$$

So we have;

$$\det(\mathbf{J}) = \frac{\pi a L}{2} \quad (25.)$$

Using Eqs. 21, 22 and 25 the element stiffness matrix, $\mathbf{K}^e$ becomes;

$$\mathbf{K}^e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{eT} \mathbf{D} \mathbf{B}^e \; \frac{\pi a L \left(a\eta + b\right)}{4} \, d\xi \, d\eta \, d\gamma \quad (26.)$$

A computer program has been developed to perform the numerical integration and calculate the element stiffness matrix.

5. Element mass matrix

The element mass matrix is defined as; [16]

$$\mathbf{M}^e = \int_{V^e} \mathbf{N}^T \rho \mathbf{N} dv \quad (27.)$$

where $\mathbf{N}$ is the shape function matrix defined in Eq. 10 and ρ is the material density. Again by using Eqs.21, 22 and 25 and the fact that the density is constant over the element, the element mass matrix becomes;

$$\mathbf{M}^e = \rho \int_{V^e} \mathbf{N}^T \mathbf{N} \; \frac{\pi a L \left(a\eta + b\right)}{4} \, d\xi \, d\eta \, d\gamma \quad (28.)$$

which is the same size as the stiffness matrix in Eq. 26.

6. Modal analysis

The equation of motion of a multi degree of freedom system without damping is of the form;

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \quad (29.)$$

where $\mathbf{x}$ is the degree of freedom vector and $\mathbf{M}$ and $\mathbf{K}$ are respectively, the system mass and stiffness matrices. When vibrating in one of the mode shapes, φ_i, all the points in the system undergo simple harmonic motion with the corresponding natural frequency ω_i, which can be stated as;

$$\mathbf{x} = \mathbf{X}_i \sin\left(\omega_i t\right) \quad (30.)$$

in which $\mathbf{X}_i$ is the amplitude vector with each component corresponding to the specific degree of freedom. Substituting Eq. 30 into Eq. 29 yields;

$$\left(-\mathbf{M}\omega_i^2+\mathbf{K}\right)\mathbf{X_i}=\mathbf{0} \quad (31.)$$

To avoid a nontrivial solution for the above equation, it follows that the determinant of the coefficient matrix, $-\mathbf{M}\omega_i+\mathbf{K}$, vanishes.

$$\left|-\mathbf{M}\omega_i^2+\mathbf{K}\right|=\mathbf{0} \quad (32.)$$

By pre-multiplying the above equation with $\mathbf{M}^{-1}$ one obtains;

$$\left|\mathbf{A}-\mathbf{I}\lambda_i\right|=\mathbf{0} \quad (33.)$$

where $\mathbf{A}=\mathbf{M}^{-1}\times\mathbf{K}$ is the *dynamic matrix* and $\lambda_i=\omega_i^2$. Solving Eq. 33 actually leads the eigenvalues, $\lambda_i, i=1,\ldots,S$, of the dynamic matrix, where S is the size of the mass and stiffness matrices which equals the number of degrees of freedom of the system. In fact the eigenvalue problem in Eq. 33 leads the square of natural frequencies as the eigenvalues of $\mathbf{A}$ which can be used to obtain the mode shapes as the eigenvectors of $\mathbf{A}$ with the aid of Eq. 31.

7. Sample problems

The modal analysis of several hollow cylinders with different boundary conditions is investigated here and the results are compared with the conventional finite element.

7.1. Example 1

Consider a hollow cylinder with length $2L=0.6$, inner radius $r_1=0.05$ and outer radius $r_2=0.06$ with modulus of elasticity $E=2\times10^{11}$ and Poisson's ratio $\nu=0.3$ and density of $\rho=7800$. The cylinder is meshed using 15 cylindrical super elements in the axial direction as shown in Fig. 2. The boundary condition is fixed at one end and free at the other end of the cylinder. The natural frequencies with super element method and conventional finite element method and their relative error are listed in table 1 for basic mode shapes.

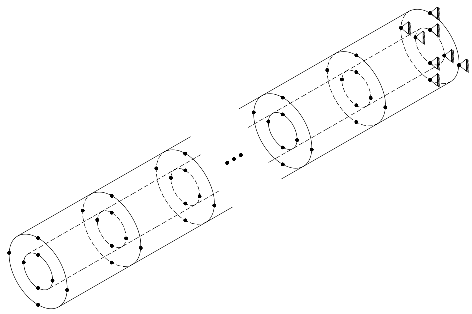

Fig. 2. Super element assembling and boundary condition

Table 1
Comparison of natural frequencies obtained from super element and conventional brick element for different modes using 15 super elements and 320 brick elements for cantilever cylinder

Mode number	**Mode shape**	**Natural frequency (Hz)**		**Relative error**
		Super element	Brick element	
1	bending	312	294	6 %
2	torsional	1339	1310	2 %
3	bending	1548	1474	5 %
4	axial	2174	2128	2 %
5	bending	3532	3395	4 %
6	torsional	4032	3962	1.7 %

Mesh refinement can significantly decrease the relative error. The results for the same cylinder are repeated in table 2 with twice the number of elements.

Table 2
Comparison of natural frequencies obtained from super element and conventional brick element for different modes using 30 super elements and 320 brick elements for cantilever cylinder

Mode number	**Mode shape**	**Natural frequency (Hz)**		**Relative error**
		Super element	Brick element	
1	bending	300	294	2 %
2	torsional	1323	1310	0.9 %
3	bending	1485	1474	0.7 %
4	axial	2142	2128	0.6 %
5	bending	3359	3395	0.1 %
6	torsional	3973	3962	0.3 %

7.2. Example 2

Consider the same cylinder in example 1 having the free-free boundary condition and meshed using 30 super elements. The first 6 natural frequencies of this cylinder vanish which correspond to the rigid body modes in the 3D space. The other natural frequencies and comparisons are listed in table 3.

Table 3
Comparison of natural frequencies obtained from super element and conventional brick element for different modes using 30 super elements and 960 brick elements for free-free cylinder

Mode number	**Mode shape**	**Natural frequency (Hz)**		**Relative error**
		Super element	Brick element	
1	bending	1653	1643	0.6 %
2	torsional	2618	2618	almost 0 %
3	bending	3614	3591	0.6 %
4	axial	4205	4205	almost 0 %
5	torsional	5243	5244	almost 0 %
6	bending	5689	5645	0.8 %

8. Conclusion

This paper has presented the development of a newly designed cylindrical super element. Using super element method it is possible to predict natural frequencies and mode shapes in cylindrical structures subjected to various boundary conditions in a precise and fast manner. Examples of these structures are abundant in any rotary system. In the case of hollow cylinders where sometimes a huge number of conventional elements are required, only few cylindrical super elements can perform the analysis with the desired accuracy. The super element method also decreases the amount of required memory space. The method is efficient in saving time and computational costs.

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