SUB-GRAPHS OF COMPLETE GRAPH

Bichitra Kalita,
Department of Computer Application,
Assam Engineering College , Guwahati-781013, Asom , India.

Abstract*: In the complete graph K_{2m+3} for $m \geq 2$, we study some structures of simple non-isomorphic Hamiltonian sub-graphs of the form H (2m+3 , 6m+3) for $m \geq 2$. The various structures of the form H(2m+3, 6m+3) for $m \geq 2$ are found and some of them are observed to give some forms of metal atom cluster compound of chemistry . Finally, we establish an algorithm to solve the traveling salesman problem when the weights of edges of the complete graph K_{2m+3} for $m \geq 2$ are non-repeated in the case of symmetrical problems. The applications of the sub-graphs H (2m+3,6m+3) for $m \geq 2$ are shown in the solution of traveling salesman problem .*

Keywords- Complete graph, Hamiltonian simple sub-graph, Non-isomorphic graph, cluster compounds, Tsp .

1. Introduction

The traveling salesman problem is an NP-Complete problem .It is a problem, where one has to study the Hamiltonian circuit with minimum expenditure (least cost route) from the weighted graph which is obtained from the cost matrix of the traveler. This problem is not yet completely solved till date. Some heuristic methods /algorithms have been considered besides the closet insertion and two optimal methods. Further, the Hungarian method is also used to solve this problem. But, it is laborious and tedious. Kruskal [1] has forwarded the application of spanning sub-tree of a graph for the solution of traveling salesman problem. Some works have been done in various parts of the world, relating to Hamiltonian graphs and traveling salesman problem by Camion [2], Ore [3] , Lin [4] ,Deo & Hakim [5] , Bellmore & Nemhauser [6] and Grinberg [7] . The edge-disjoint Hamiltonian circuits of the complete graph K_{2m+2} for $m \geq 2$ has been discussed by Kalita [8]. Again Burkand. et al [9] discussed the traveling salesman problem with the help of permuted Monge matrices. Recently, Kalita [10] has put forward a Heuristic method of traveling salesman problem under different conditions for the complete graph K_{2m+2} for $m \geq 2$. An up-to date algorithm /method has been forwarded by Kalita [11] and by this method, the minimum cost route of a traveler for complete weighted graph, when the weights are non-repeated has been determined. This method was comparatively an efficient method than the other existing methods.

The very interesting non-isomorphic Hamiltonian sub-graphs of the form H (2m+2,6m) for $m \geq 2$, obtained from the complete graph K_{2m+2} for $m \geq 2$ have been studied by Kalita [12], where nice application of such type of non-isomorphic Hamiltonian sub-graphs are found in case of traveling salesman problem . It was found that some structures of metal atom cluster compounds of chemistry have been found from the structures of non-isomorphic Hamiltonian simple sub-graphs of the complete graph K_{2m+2} for $m \geq 2$.

The paper is organized as follows:

The section 1 includes the introduction part containing the works of other researchers of the world. Section 2 includes notations and terminologies .The theoretical investigations and discussions of non-isomorphic Hamiltonian simple Sub-graphs of the complete graph K_{2m+3} for $m \geq 2$ is included in section 3 .In section 4 of the paper, we have focused some new direction about the structures of metal atom cluster compounds of chemistry and they are found from the structures of non-isomorphic Hamiltonian simple sub-graphs of the complete graph K_{2m+3} for $m \geq 2$.

In section 5 , an algorithm has been developed to study the solution of traveling salesman problem with the help of some non-isomorphic Hamiltonian sub-graphs of the complete graph K_{2m+3} for $m \geq 2$. We have studied the traveling salesman problem, considering the complete graph K_{2m+3} for $m \geq 2$ as weighted graph. In section 6, conclusion is included.

2. Notation and Terminology

The notations and terminologies are considered from the standard references [1-9] .The complete graph K_{2m+3} for $m \geq 2$ is considered to find out the non- isomorphic Hamiltonian sub-graphs of the form H (2m+3 , 6m+3) for $m \geq 2$ .The maximum and minimum degree of a vertex is denoted by Δ and ∂ respectively . The cost matrix of a traveler is always considered for odd number of cities.

3. Theoretical investigation and structure of sub-graph

Let us consider the complete graph K $_{2m+3}$ for $m \geq 2$ .It is known that there are $2m^2 +5m+3$ edges for $m \geq 2$ in the complete graph K_{2m+3}. Now deleting 2m+2 edges for $m \geq 2$ from the complete graph K_{2m+3} , keeping in mind that the sub-graph thus obtained after deletion of the edges must have minimum degree at least $\partial = 2$ and maximum degree $\Delta = 2m+ 2$ for $m \geq 2$. It is found that the structure of this type of sub-graph of the complete graph K_{2m+3} takes the form H (2m+3 ,6m+3) for $m \geq 2$. Obviously, this sub-graph is a Hamiltonian graph as there exists at least one Hamiltonian circuit connecting all vertices of the graph .This graph has at least one vertex with minimum degree $\partial = 2$ and maximum degree Δ =2m+2 for $m \geq 2$ and the degrees of all other vertices lie between 2 and 2m+2 for $m \geq 2$. Now our point is to find out how many sub-graphs there are in the complete graph K_{2m+3} which are of the forms H(2m+3, 6m+3) for $m \geq 2$ subject to the condition that the degrees of the vertices lie between 2 and 2m+2 for $m \geq 2$ .But, some structures of metal atom cluster compounds are found in these sub-graphs (later it will be discussed) .The following theorem gives the number of non-isomorphic Hamiltonian sub-graphs of the form H (2m+3 ,6m+3) for $m \geq 2$ of the complete graph $K_{2m+3.}$

3.1 .Theorem : The number of non-isomorphic Hamiltonian simple sub-graphs of the form H(2m+3,6m+3) for $m \geq 2$ of the complete graph K_{2m+3} is 5n+8 for $n \geq 1$ with simultaneous changes of $m \geq 2$.

Proof :It is clear that the existence of the sub-graph H(2m+3 , 6m+3) for $m \geq 2$ from the complete graph K_{2m+3} for $m \geq 2$ is possible as the 6m+3 edges for $m \geq 2$ together with the deleted edges (2m+ 2)of the complete graph K_{2m+3} for $m \geq 2$ gives 8m+5 edges. There is one-one correspondence between the edges 8m+5 and the edges $2m^2 +5m+3$ of the complete graph K_{2m+3} for $m \geq 2$. Again it is found that all the sub-graphs of the form H (2m+3 , 6m+3) for $m \geq 2$ are of different degrees of the vertices .(only two graphs are given below by the figure –1& 2. for m=2)

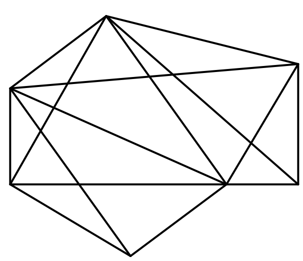

Figure-1

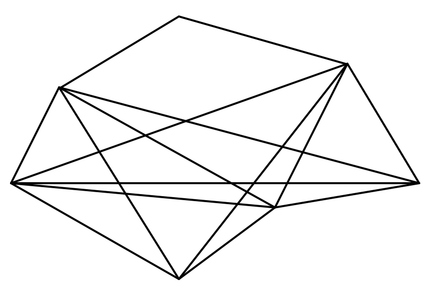

Figure-2.

Hence, they (all the sub-graphs of the form H (2m+3,6m+3) for $m \geq 2$) are non- isomorphic as it is found that the incidence relationship between the vertices of the graphs are not equal (that is the sub-graphs have same number of vertices and same number of edges but not have the same number of vertices with given degrees). Now , we shall show that these sub-graphs H (2m+3 , 6m+3) for $m \geq 2$ are Hamiltonian .It is known that the complete graph K_{2m+3} is Hamiltonian and it is regular of degree 2m+2 for $m \geq 2$.The sub-graph H(2m+3, 6m+3) for $m \geq 2$ of the complete graph K_{2m+3} has at least one vertex of degree (∂ =2) and all the degrees of the other vertices lie between 2 and 2m+2 for $m \geq 2$. It can be shown that there are thirteen graphs of the form H (7 , 15) for m=2 and all of them are Hamiltonian (Figure 2 above is one of them) as it has at least one Hamiltonian circuit . Since the degrees of all other vertices of the sub-graphs H (2m+3, 6m+3) for $m \geq 2$ lie between 2 and 2m+2 for $m \geq 2$ and each of them contains at least one Hamiltonian circuit and therefore they are Hamiltonian .Again the sub-graphs do not contain any self loop or parallel edges and so they are simple .

Finally, we show that there are actually 5n+8 numbers of Hamiltonian simple sub-graphs for $n \geq 1$ with simultaneous changes of $m \geq 2$ of the complete graph K_{2m+3}.

We prove it by induction method. It can be proved from the construction process of the sub-graph H (2m+3, 6m+3) for m= 2, deleting 2m+2 = 6 edges from the complete graph K_7 that there are 5.1+8 =13 sub-graphs for n=1. These 13 sub-graphs are Hamiltonian simple and non-isomorphic .Therefore, the result is true for n=1 with simultaneous changes of m =2. Let the result be true for n= k-1 with the simultaneous changes of m=k .Hence there are 5(k-1) +8 = 5k+3 non-isomorphic Hamiltonian simple sub-graphs .Now we shall show that the theorem is true for n=k with simultaneous changes of m=k+1.

The number of non-isomorphic Hamiltonian simple sub-graphs is 5(k) +8 when we consider the value m=k+1. Since according to the statement of the theorem we need to consider the value of $n \geq 1$, so we must take $n=k \geq 1$ on $m=k+1 \geq 2$ i.e. $m=k \geq 1$, which shows that the theorem is true for any value of $n \geq 1$ with simultaneous changes of the values of $m \geq 2$. This is the proof of the theorem.

4. New direction:

It is found from the construction process of the non-isomorphic simple Hamiltonian graphs H(2m+3,6m+3) for $m \geq 2$ out of the complete K_{2m+3} for $m \geq 2$ gives some new idea about the structures of metal atom cluster compound of chemistry. The structure of $Os_7 (CO)_{21}$ already observed by the chemists and the structure thus found is shown in figure –3 .

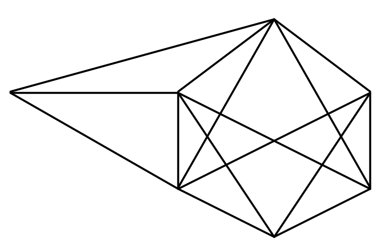

Figure –3 .

The structure shown above is one of the Simple non-isomorphic Hamiltonian sub-graphs out of 5n + 8 =13 for n=1 with simultaneous changes of m=2. The theorem 3.1 above gives this number of non-isomorphic simple Hamiltonian sub-graphs of the complete graph K_{2m+3} for $m \geq 2$. This theorem may give some structures of metal atom cluster compounds (claimed) for m=3 also. The following structure (figure-4) is found for m =3.

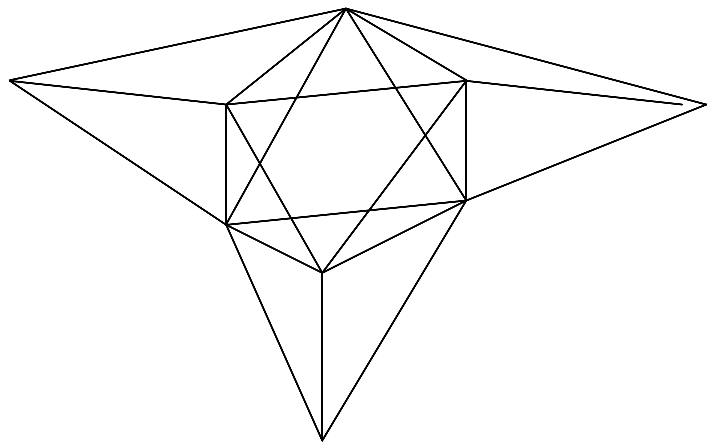

Figure –4

Chemists accepted it (the above structure of figure-4) but the structure till date is not tested [16-19]. We have the figure-3 (already known) and figure-4 (not tested) are the structures of the graph of the form H (7, 15) for m=2 and H (9, 21) for m =3 which are obtained from the theorem 3.1 above. Hence we intend to claim that these types of structures of various kinds of metal atom cluster compound can be found from the theorem for the values of $m \geq 4$.The various properties of these types of metal atom cluster compounds relating to the total electron count etc. of carbon atom would be found .

In this context I would like to invite the chemists to observe and examine the existence of such structures found by the theorem 3.1.

Some of the structures included in the sub-graphs given by theorem 3.1 may also be explained through traveling salesman problem described in the algorithm.

5. Algorithm for symmetrical problems:

The distance between every pair of vertices/cities is independent of direction of the journey

The complete graph K_{2m+3} for $m \geq 2$ considered here, is a complete weighted graph where the weights are non repeated .Different cases are considered .

Case 1: In case of 2m+2 number of consecutive greatest weights for $m \geq 2$ forming a covering or incident with a vertex of the complete graph K_{2m+3} for $m \geq 2$ formed from the cost matrix of the traveler, we consider the following steps.

Step 1: Delete the greatest 2m weights for $m \geq 2$ from the vertex(v) where the other two greatest weights are incident with v.

Step 2: Construct the graph H (2m+3 , 6m+3) for $m \geq 2$ with minimum degree of the vertex v (∂ =2). (This graph is one of the sub-graphs of K_{2m+3} for $m \geq 2$).

Step 3: Form the n x n reduced matrix from the graph obtained from the graph discussed in step 2.

Step 4: Select any row from the matrix obtained from the step 3 and consider the assignment in the cell where the least weight (least cost) is given. The starting city/vertex will be the city/vertex of this row .

Step 5. Select the city/vertex which lies in the column where the least cost lies in the row found in step 4.

Step 6: Go to the other rows where the city already considered from the column of step 5 lies, and select the least cost of the row and simultaneously select the City in such a way that no city is over lapped, as discussed in steps 4 and step 5.

Step 7: Continue the selection procedure of the row / column till the starting city/vertex considered in step 4 is reached.

Step 8: Stop.

5.1. Experimental result of case 1

The following cost matrix (table-1) is considered for a traveler for seven cities (that is for m =2) .

Table-1

	A	B	C	D	E	F	G
A	-	37	38	40	60	55	42
B	37	-	24	25	10	7	36
C	38	24	-	29	30.5	26	12
D	40	25	29	-	5	15	35
E	60	10	30.5	5	-	4.7	22
F	55	7	26	15	4.7	-	20
G	42	36	12	35	22	20	-

Solution: After canceling the four consecutive greatest weights that is 60 ,55, 42 and 40 (as discussed in step 1 of the algorithm) and obtaining the reduced matrix of the graph obtained after step 1, we first consider the first row and second column with assignment 37 . That is, we can start journey from city A to city B and this gives the step 4 and step 5.

Now, continuing other steps from step 6 to step 8 of the algorithm one after another we have the least cost route with cost 137 and the route will be as shown below.

A→ B→E→D→F→G→C→A.(that is the traveler will start his journey from the city A and he will end his journey at A covering all cities with a minimum cost).

Case 2: When the covering contains all the consecutive greatest 2m+2 for $m \geq 2$ but it is not incident with a vertex .Then the following steps are considered.

Step1: Delete all the greatest 2m+2 consecutive weighted edges for $m \geq 2$ from the complete graph K_{2m+3} for $m \geq 2$.

The other steps are same as discussed in the case1 of the above algorithm.

Here, the graph found from the step 2 of case1 may not be the graph of the type having one vertex of minimum degree $\partial=2$..

Case 3: If the covering contains minimum consecutive 2m+2 smallest weights with a vertex for $m \geq 2$,then the following steps are considered.

Step 1: Delete the 2m consecutive smallest weights for $m \geq 2$ keeping the smallest two weights out of 2m+2 for $m \geq 2$ incident with the vertex.

Step 2: Construct the graph H(2m+3, 6m +3) for $m \geq 2$ after deletion of 2m edges of Step1 .This graph will be a graph of minimum degree $\partial = 2$ incident with the vertex v.

Step 3: Find the Hamiltonian circuit of the graph obtained from step2, considering two edges of smallest weight of the graph with other four edges of smallest weights.

Step 4: The Hamiltonian circuit of step 3 is the least Hamiltonian circuit.

Step 5: stop

5.2. Experimental result of case 3

The following cost matrix (table-2) is considered for seven Cities.

Table-2

	A	B	C	D	E	F	G
A	-	1	2	3	4	5	6
B	1	-	21	7	10	12	8
C	2	21	-	20	14	9	15
D	3	7	20	-	11	13	19
E	4	10	14	11	-	18	16
F	5	12	9	13	18	-	17
G	6	8	15	19	16	17	-

Solution: Applying the algorithm of case3 we have the least cost route as B→ A→ C→E→G→ F→D→ B with total least cost 70 for the journey.

6. Conclusion

The algorithm discussed above gives some applications of the non- isomorphic Hamiltonian sub-graphs of the complete graph K_{2m+3} for $m \geq 2$ for the solution of the traveling salesman problem for certain cases . The structures of the sub-graphs discussed in the theorem 3.1 may lead to new direction as claimed above in the study of the structures of metal atom cluster compound of chemistry in near future.

References:

[1] Kruskal, J.B., "On the shortest spanning sub-tree of a graph and traveling salesman problem". Proc. Amer. Math. Soc 7. pp.48-50, 1956

[2] Camion, P. "Chemie.et circuits Hamiltonian des graphe complets" C.R. Acad. Sci. Paris 249.pp. 2151-2152.1959.

[3] Ore, O. "Notes on Hamiltonian Circuits" Amer. Maths. Monthly. pp.55.1960.

[4] Lin, S. "Computer Solution of traveling salesman problems" BSTJ,44.pp.2245-2269, 1965.

[5] Deo, N & Hakim, S. L. "The shortest generalized Hamiltonian tree" Third Annual allerton Conference, University of Illinois.pp.879-888, 1965.

[6] Bellmore, M. & Nemhauser, G .L. "The traveling salesman problems" A Survey, O.R. 16. pp.538-558, 1968.

[7] Grinberg, E .J. "Planar homogeneous graphs of degree three without Hamiltonian circuits". Latvian . Math year book. 4 . pp.51-58.1968.

[8] Kalita, B. "SCN with DCS tree with cover and Hamiltonian circuits" Proc. International. Workshop on Telematics. 1-15 June , pp.163-168, 1995.

[9] Burkand, R .E., Deineko, V.G., Woeginger, G.T., “The traveling salesman problems on permuted Monge Matrices”. J. Com. Optimization, Vol. 24 pp.333-350., 1999.
[10] Kalita, B.“Some Investigations on Graph Theory. Ph.D. thesis” Finance India, Vol.xix No 4, December , pp. 1430-1438 ,2005.
[11] Kalita , B. “Application of three edged extension of planar (TEEP) graph for the solution of TSP”. Proc. 48th ISTAM Congress 2003.
[12] Kalita, B. “Non- Isomorphic Hamiltonian Sun-graphs of complete graph and their application” Proc. ISCA .Ahmedabad .2005.
[13] Kalita, B. “Construction of Strongly regular graph from TEEP graphs” Bull.of Pure & Applied Sc. Vol. 22E. N0-1 pp.113-117., 2003.
[14] Kalita, B. “A new set of non-planar graphs” . Bull. of Pure & Applied Sc. Vol.24E .No-1 pp.29-38, 2005.
[15] Kalita, B., Baruah, H.K., Kalita, B.C. “On isomorphism of graphs” Proc 42nd ISTAM Conf. Pp.179-187. 1997.
[16] D.P.M Mingos & D . J .walls, “Introduction to cluster Chemistry”. Prentice Hall, Englewood cliffs . N.J 1990
[17] R. D .Adams etal , J .Am .Chem. Soc. 116 , 9103, 1994.
[18] J.D.Corbett, J.Chem.Soc.Dalton Trans. , 575,1996.
[19] D.Corbet, J. , Alloys compounds . , 229, 1995.