On Induced Matching Partitions of Certain Interconnection Networks

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Abstract – The induced matching partition number of a graph G, denoted by imp(G), is the minimum integer k such that V(G) has a k-partition ($V_1, V_2 \ldots V_k$) *such that, for each i,* $1 \leq i \leq k$, $G[V_i]$, *the subgraph of G induced by* V_i, *is a* 1*-regular graph. The induced matching k-partition problem is NP-complete even for k = 2. In this paper we investigate the induced matching partition problem for butterfly networks. We identify hypercubes, cube-connected cycles, grids of order m x n, where at least one of m and n is even, as graphs for which imp(G) = 2. In the sequel we prove that imp(G) does not exist for grids of order m x n where m and n are both odd and Mesh of trees MT(n), n ≥ 2.*

Keywords: Matching, partition, induced graph, mesh of trees, butterfly networks

1.0 Introduction and Background

Let *G* be a graph with vertex set *V*(*G*) and edge set *E*(*G*) .We assume that *G* has no loops or multiple edges. Two edges are *independent* if they have no common endpoint, and a *matching M* in *G* is a set of (pairwise) independent edges. Two ends of an edge in *M* are said to be *matched under M.* A vertex *v* is said to be *M-saturated* if some edge of *M* is incident with *v*, otherwise *v* is *M-unsaturated.*

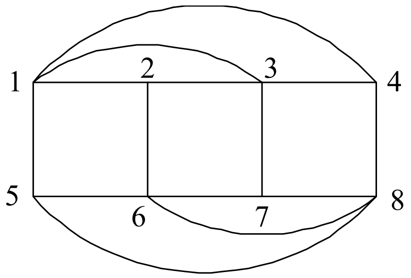

Figure 1: A graph with no induced matching partition

A matching *M* is a *perfect matching* if every vertex in *G* is an endpoint of one of the edges in *M*. Let *G* be a graph with a perfect matching. An *induced matching k-partition* of a graph *G* is a *k*-partition

(V_1, V_2 ,…, V_k) of $V(G)$ such that, for each i, $1 \le i \le k$, the subgraph $G[V_i]$ of G induced by V_i is 1-regular. The *induced matching partition number* of a graph G, denoted by $imp(G)$, is the minimum integer k such that G has an induced matching k-partition. The *induced matching k-partition problem* asks whether a given graph G has an induced matching k-partition or not. Terminology and notation not defined here can be found in [1, 5, 6].

The example in Figure 1 illustrates that not every perfect matching of a graph G induces $k = 2$. Let $M_1 = \{(1, 2), (5, 6), (3, 4), (7, 8)\}$, $V = V_1 \cup V_2$ where $V_1 = \{1, 2, 7, 8\}$ and $V_2 = \{3, 4, 5, 6\}$. Here $k = 2$. Again let $M_2 = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$. This is a perfect matching but this does not induce $k = 2$.

The induced matching k-partition problem was first studied as a combinatorial optimization problem [4]. If a graph G has a perfect matching then $imp\ (G) \le 2\ \Delta(G) - 1$ and $imp(G) = 2\Delta(G) - 1$ if and only if G is isomorphic to either K_2 or C_{4k+2} or the well-known Petersen graph, where C_n is the cycle of length n [9]. The induced matching k-partition problem is NP-complete, and also NP-complete for $k = 2$ and for 3-regular planar graphs, respectively [4, 6]. In this paper we study the induced matching partition problem for the butterfly networks. We also identify hypercubes, cube-connected cycles, and grids of order m x n, where at least one of m and n is even as graphs for which $k = 2$. In the sequel we prove that $imp(G)$ does not exist for grids of order m x n where m, n are both odd and Mesh of trees $MT(n)$, $n \ge 2$.

2.0 Main Results

The following theorem gives a forbidden subgraph for a graph to have a perfect matching.

Theorem 1: Let G be a graph containing the graph H shown in Figure 2 as an induced subgraph where $\underset{G}{\deg}\ v_2 = \underset{G}{\deg}\ v_3 = \underset{G}{\deg}\ v_5 = \underset{G}{\deg}\ v_6 = 2$, $\underset{G}{\deg}\ v_4 = 4$, $\underset{G}{\deg}\ v_1 \ge 2$, and $\underset{G}{\deg}\ v_7 \ge 2$. Then, G does not have a perfect matching.

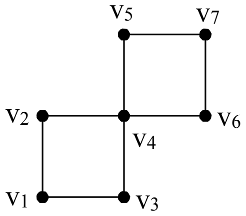

Figure 2: A forbidden graph H

Proof: Assume on the contrary, that G has a perfect matching M. The following cases arise.

Case 1: v_1 is saturated by an edge in M with its other end in $V(G) \setminus V(H)$ .Since v_2 is of degree 2, the edge $v_2 v_4$ is an edge in M. But then, v_3 is left out and so G does not have a perfect matching.

Case 2: $v_1 v_2$ is a member of the matching. Since v_3 is of degree 2 and v_1 is already saturated, $v_4 v_3$ is also a member of the matching. Again, v_5 is of degree 2 and v_4 is saturated. Therefore, $v_5 v_7$ has to be a member of the matching. Now v_6 is left out and hence G does not have a perfect matching. □

Theorem 2: Let G contain an even cycle C as an induced subgraph with alternate vertices of degree 2 and at least one of the remaining cycle vertices of degree greater than 2. If G has a perfect matching M then no edge in M saturates a vertex of degree > 2 on the cycle with its other end in $V(G) \setminus V(C)$.

Proof: Assume the contrary. Let $C = v_1v_2 \dots v_{2r}v_1$. If deg $v_3 > 2$ and is saturated by an edge $e = v_3y \in M$ with $y \notin V(C)$, then v_2 will be left unsaturated. Hence $v_2v_3, v_4v_5, \dots, v_{2r-2}v_{2r-1} \in M$. But then the vertex v_{2r} which is of degree 2 remains unsaturated. □

The above theorem can also be stated as follows.

Theorem 3: Let *G* contain an even cycle as an induced subgraph with alternate vertices of degree 2 and at least one of the remaining cycle vertices of degree greater than 2. If *G* has a perfect matching *M* then *M* contains alternate edges of *C*. □

Notation: We call a cycle described in Theorem 2 as a cycle of Type α .

3.0 The Induced Matching Partition Number for Certain Interconnection Networks

In this section we study the induced matching partition problem for certain interconnection networks such as butterfly networks, hypercubes, cube-connected cycles, grids and Mesh of trees For definition and properties of the networks, we refer to [6].

3.1 Butterfly Networks

Here is the mathematical definition of a butterfly architecture which is denoted by *BF*(*V*, *E*) or *BF*(*n*). The set *V* of nodes correspond to pairs $< w, i >$, where *i* is the dimension or level of the node ($0 \leq i \leq n$) and w is a *n*-bit binary number that denotes the row of the node. Two nodes $< w, i >$ and $< w', i' >$ are linked by an edge if and only if $i' = i + 1$ and either:

1. *w* and *w'* are identical, or
2. *w* and *w'* differ in precisely the i^{th} bit.

The butterfly network *BF*(*n*) has $(n + 1)\ 2^n$ vertices. An easy consequence of Theorem 1 is that *BF*(2) has no perfect matching.

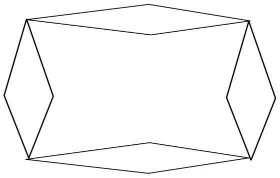

Figure 3: Butterfly BF(2)

Theorem 4: For butterfly *BF*(*3*), *imp*(*BF*(*3*)) *=2.* □

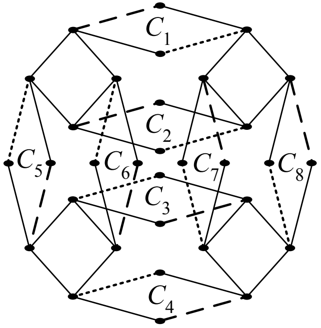

Figure 4: Butterfly BF(3)

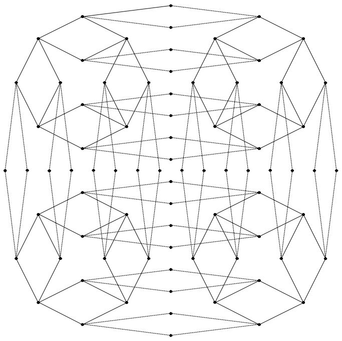

Figure 5: Butterfly BF(4)

Proof: The butterfly on 32 vertices has eight 4-cycles $C_1, C_2, ..., C_8$ of Type α and the vertex sets of these eight 4-cycles effect a partition of *V*(*G*). See Figure 4.By Theorem 3 alternate edges in these cycles of Type α should be in any perfect matching of *BF*(3). Again if the subgraph induced by the vertex sets of any two 4 – cycles, say C_1 and C_5, of Type α is connected then the induced subgraph is nothing but the cycles C_1 and C_5 together with a bridge between them. Hence if $e = xy$ is a bridge between C_1 and C_5 then the end vertex of the edge in C_1 with one end as *x* and the end vertex of the edge of C_5 with one end as *y* should be placed in different partitions V_1 and V_2 respectively. Thus V_1 and V_2 induce a matching partition of *V*(*G*).This implies that *imp*(*BF*(*3*)) =*2.* □

Theorem 5: Perfect matching does not exist for *BF*(4).
Proof: There are *sixteen* 4-cycles of type α in *BF*(4) (cycles in broken lines). See Figure 5. By Theorem 3, if there exists a perfect matching *M*, then *M* contains alternate edges of each 4-cycle of Type α. Deletion of vertices of degree 2 in these cycles results in the disconnection of the graph yielding 4 components C_1, C_2, C_3 and C_4, each isomorphic to *BF*(2). See Figure 6. Consider one of the components C_1. All the vertices of degree 2 in this component are already saturated by edges in *M*. Take a vertex *v* of degree 4 in this component. All the four vertices adjacent to *v* are already saturated. Hence *v* remains unsaturated. This is true for every vertex of degree 4 in the four components. Thus *BF(4)* has no perfect matching. □

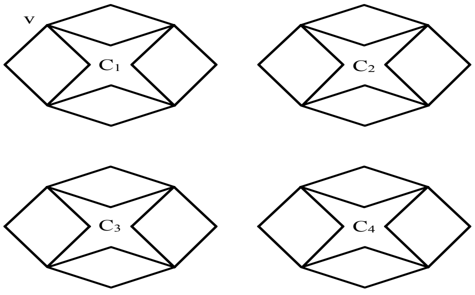

Figure 6: Components C*ᵢ*, 1≤ *i* ≤ 4

Theorem 6: Let *BF*(*n*) be the butterfly network on $(n+1)2^n$, $n \geq 2$ vertices. Then *imp*(*BF*(*n*)) = 2 if *n* is odd and *imp*(*BF*(*n*)) does not exist when *n* is even.
Proof: We prove the result by induction on *n*. As the base case we have already proved in Theorem 4 that *imp*(*BF*(3)) = 2 and in Theorem 5 that perfect matching does not exist for *BF*(4). Assume the result to be true for all butterfly networks when *n* is less than or equal to $2m - 1$ *or* $2m - 2, m \geq 3$.

Now, *BF*(*n*), $n \geq 8$, contains 2^n cycles of Type α. By Theorem 3, alternate edges in each of the cycles should be present in any perfect matching *M* of *BF*(*n*). Deletion of vertices of degree 2 in these cycles results in the disconnection of the graph yielding 4 components, each isomorphic to *BF*(*n* – 2*)*. The vertices of degree 2 in the 2^{n-2} cycles of Type α in each *BF*(*n* – 2*)* are already *M*-saturated. Hence none of the edges in these 4x2^{n-2} cycles of Type α is in *M.* Further deletion of vertices of degree 2 in the four components results in the number of components increasing to 16. Each of these 16 components is isomorphic to *BF*(*n* – 4*)* .

If *n* = *2m* – 1 then by induction hypothesis *imp*(*BF*(*n* – 4)) = *2.* Now color the edges in *M* with colors red and blue such that the edges in the 4 components of *BF*(*n* – 2) which are mirror images along the imaginary lines joining vertices of degree 2 in *BF*(*n* –2) are colored with complementary colors. Again color the 4 components of *BF*(*n*) which are mirror images along the imaginary lines joining vertices of degree 2 in *BF(n)* are colored with complementary colors. If the graph induced by vertices of a cycle of Type α in *BF*(*n*) and a cycle of Type α in *BF*(*n*-4) is connected then it is nothing but the two cycles together with a bridge *xy* between the two. If *y* is in *BF*(n – 4) and is saturated by an edge in *M* of color red then color the edge in the induced subgraph that saturates *x* as blue and conversely. Thus we get *imp*(*BF*(*n*)) = 2.

On the other hand if $n = 2m$ then by induction hypothesis *imp*(*BF*(n – 4)) does not exist. Hence following the arguments as in the proof of Theorem 5 we see that a perfect matching does not exist for *BF*(*n*). □

3.2 Hypercubes and Cube-Connected Cycles

The *n-dimensional binary hypercube*, or an *n-cube,* is a graph with 2^n vertices labeled by *n*-bit binary strings, with edges joining two vertices whenever their labels differ in a single bit [5, 6].

Theorem 7: For the Hypercube Q^n, *imp* (Q^n) = 2, $n \geq 2$.

Proof: The proof is by induction on *n*. When $n = 2$, Q^2 is a 4-cycle say a_1, a_2, a_3, a_4. Let $X = \{a_1, a_2\}$ and $Y = \{a_3, a_4\}$. Then *X* and *Y* form an induced matching partition of the vertex set of Q^1 proving that *imp* (Q^1) =2. Assume the result to be true for Q^{n-1}. Consider Q^n. Let (a_i, b_i), $i = 1,2,..., 2^{n-1}$, be the edges in Q^n whose removal disconnects Q^n into two (n – 1)-dimensional hypercubes Q_1^{n-1} and Q_2^{n-1} where $V(Q_1^{n-1}) = \{a_1, a_2 \dots a_{2^{n-1}}\}$ and $V(Q_2^{n-1}) = \{b_1, b_2 \dots b_{2^{n-1}}\}$. By induction hypothesis *imp*(Q^{n-1}) = 2. Let *A* and *B* be induced matching partitions of $V(Q_1^{n-1})$ and $V(Q_1^{n-2})$ respectively. Consider the edges (a_i, b_i) and (a_j, b_j). If the subgraph induced by a_i, b_i, a_j, b_j is a 4-cycle and if (a_i, a_j) is in *A*, then place (b_i, b_j) in *B*. This gives *imp*(Q^n) = 2. □

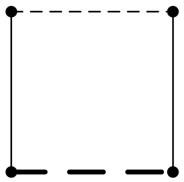

Figure 7: Hypercube Q^2

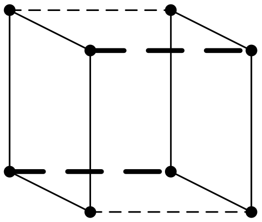

Figure 8: Hypercube Q^4

A Cube-connected cycle CCC(*n*) is a graph obtained from the hypercube Q^n on 2^n vertices by replacing each vertex by an *n*-cycle [5, 6].

The proof of the following theorem is similar to Theorem 7.

Theorem 8: For the Cube Connected Cycle *CCC*(*n*), *imp* (*CCC*(*n*)) = 2. □

3.3 Grids

Theorem 9: Let *G* be a grid of order *m x n* where *m* and *n* are both not odd. Then *imp*(*G*) = 2.
Proof: Let the *mn* vertices of the grid be labeled by the corresponding co-ordinate positions (*i*, *j*), $1 \leq i \leq m, 1 \leq j \leq n$. We assume that $m \leq n$. Let *A* be the edges

$((2i-1, 4j-3), (2i-1, 4j-2))$ for $1 \le i \le \lceil m/2 \rceil, 1 \le j \le \lceil n/4 \rceil$ and $((2i, 4j-1), (2i, 4j))$ for $1 \le i \le \lfloor m/2 \rfloor$, $1 \le j \le \lfloor n/4 \rfloor$, shown in broken lines in Figures 9 and 10. Again let *B* be the edges $((2i-1, 4j-1), (2i-1, 4j))$, for $1 \le i \le \lceil m/2 \rceil, 1 \le j \le \lceil n/4 \rceil$ and $((2i, 4j-3), (2i, 4j-2))$, $1 \le i \le \lfloor m/2 \rfloor$, $1 \le j \le \lfloor n/4 \rfloor$, shown in bold lines in Figures 9 and 10. Clearly vertices in *A* and *B* induce a matching partition of *V* into V_1 and V_2. Thus *imp*(*G*) = 2.□

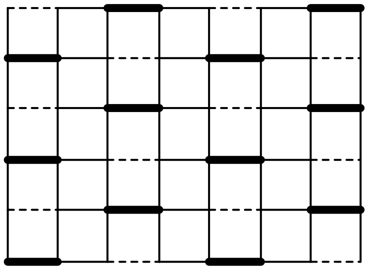

Figure 9: *m* x *n* Grid, *m* and *n* are even

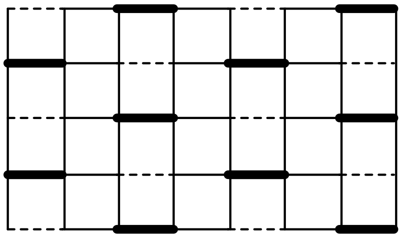

Figure 10: *m* x *n* Grid, *m* odd and *n* even

Remark: If *m* and *n* are odd, an $m \times n$ grid has odd number of vertices and hence a perfect matching does not exist.

3.4 Mesh of trees

The $N \times N$ mesh of trees is constructed from an $N \times N$ grid of processors by adding processors and wires to form a complete binary tree in each row and each column. The leaves of the trees are precisely the original N^2 nodes of the grid, and the added nodes are precisely the internal nodes of the trees. Overall the network has $3N^2 - 2N$ processors. The leaf and the root processors have degree 2, and all other processors have degree 3. Let *MT*(*n*) denote the $2^n \times 2^n$ mesh of trees. See Figure 11.

Theorem 10: Mesh of trees *MT*(*n*), $n \ge 2$ does not have a perfect matching.

Proof: *MT*(*n*), $n \ge 2$ contains 2^{2n-2} 8-cycles of Type α. Hence by Theorem 3 alternate edges on these 8-cycles are present in any perfect matching in *MT*(*n*). Deletion of the vertices of these 2^{2n-2} 8-cycles of Type α yields 2^{n+1} components, each of which is isomorphic to *BT*($n-1$) where *BT*(*n*) is a complete binary tree on $2^n - 1$ vertices. Since *BT*($n-1$) has odd number of vertices it has no perfect matching. Thus *MT*(*n*), $n \ge 2$ has no perfect matching. □

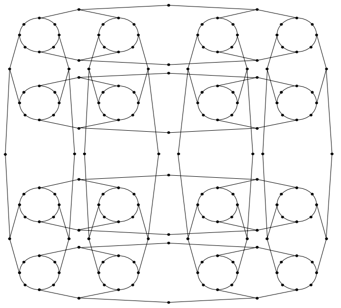

Figure 11: An 8x8 Mesh of Trees *MT*(3)

4.0 Conclusion

In this paper, induced matching partitions of Butterfly Networks have been determined. Architectures such as hypercubes, cube connected cycles, grids have been proved to have induced partition number equal to 2. As the induced matching *k*-partition problem is *NP*-complete for $k = 2$, it would be interesting to identify other interconnection networks for which $k = 2$. It would also be interesting to consider interconnection networks for which $k > 2$. □

5.0 References

[1] J.A. Bondy, U.S.R. Murty, "Graph Theory with Applications", Macmillan Press Ltd., London, 1976.

[2] K. Cameron, "Induced matchings", Discrete Appl. Math. 24: 97–102, 1989.

[3] R.T. Faudree, A. Gyarfas, R.M. Schelp, Z. Tuza, "Induced matchings in bipartite graphs", Discrete Math. 78: 83–87, 1989.

[4] M.R. Garey, D.S. Johnson, "Computers and Intractability: A Guide to the Theory of NP-Completeness", Freeman, San Francisco, CA, 1979.

[5] Jumning Xu, "Topological Structure and Analysis of Interconnection Networks", Kluwer Academic Publishers, ISBN 1-4020-0020-0, 2001.

[6] T. F. Leighton. "Introduction to parallel algorithms and architecture: Arrays, Trees, Hypercubes", Morgan Kaufmann Publishers, San Mateo, California, 1992.

[7] L.Lovasz, M.D. Plummer, "Matching Theory", Elsevier, North-Holland, Amsterdam, 1985.

[8] T.J. Schaefer, "The complexity of satisfiability problems", Proceedings of the 10th Annual ACM Symposium on Theory of Computing, Association for Computing Machinery, New York, pp. 216–226, 1978.

[9] J.Yuan, Q.Wang, "Partition the vertices of a graph into induced matching", Discrete Math., 263: 323 -329, 2003.