New Approach To A Class Of Matrices *

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Abstract

The displacement structure is extended to a Kronecker matrix $W \otimes Z$. A new class of Kronecker-like matrices with the displacement rank r, $r < n$ will be formulated and presented. The computational complexity of multiplication with vectors for Kronecker-like matrices has been accelerated. Applying the displacement, which was originally discovered by Kailath, Kung and Morf [15], a new superfast algorithm for the multiplication of a Kronecker-like matrix of the size $n1 \times n1$ over a field with a vector will be designed. The memory space cost of the number of the elements stored for a Kronecker-like matrix of the size $n1 \times n1$ over a field is $O(rn)$. The cost of the number of the arithmetic operations for the product of a Kronecker-like matrix with the displacement rank r and a vector is reduced dramatically to $O(rn)$.

Key words: Structure matrices, Matrix-vector product, Kronecker-like matrices, Superfast algorithm.

1991 Mathematics Subject Classification: 15A15, 15A33, 47B35, 65F30, 68Q25, 68W25, 68W35.

1 Introduction

In applied mathematics, computer science and engineering, computations are part of research processes. To improve efficiency of computations, we always need new techniques. Many active researchers have been developed many novel methods which have been implemented in our practical computations. For example, Kailath, Kung and Morf ([15]) have successfully exploited the displacement structure in fast computations. The displacement structure has applied in the efficient computations of many structured matrices such as Toepletz-like [20]. A variety of applications includes conformal mapping, tangential Nevanlinna-Pick interpolation, convolution, solution of integral equations, rational interpolation, Fast Fourier Transform, Fast Cosine/Sine transform, algebraic decoding, linear processing, oil exploration, polynomial interpolation, polynomial evaluation, signal and image processing (cf. [28], [9], [27], [26], [7], [24], [16], [22], [21], [3], [8], [19], [26]).

Reducing the computational complexity is one of the main goal for many active researchers. Many developments have been pro-

*Supported by PSC CUNY Awards 36-1087 and 36-494, City Tech and NSF-hrd 9703600.

posed to solve the problem of a matrix-vector product more efficiently. In general, it is well know that a matrix-vector product costs at most $O(n^2)$ operations for a general matrix of the size $n \times n$. One simply way to speed up the computing process is to observe the special properties of matrices such as Cauchy, Hankel, Toepletz, diagonal, tridiagonal and large sparse matrices over a field. For more complicated way, the displacement can be used ([10], [11]). A substantial improvement has been developed by using the displacement approach for matrices such as Cauchy-like, Hankel-like, Toepletz-like and Confluent Cauchy-like (see, [1], [14], [10], [11], [23]. One of the challenge problem is that whether the displacement approach can be extended to a new class of structured matrices.

The memory space to store the structured matrices can be saved by applying the displacement structure. The running time in computing the structured matrices can be reduced. Instead of working on the entries of a matrix itself, we may use the generators of a matrix in our computations. Many structured matrices can be re-constructed through their generators with the displacement operators (cf. [6], [10], [14], [12], [16], [17], [18], [25]).

Let recall the well known Sylvester operator, [15], [10], [11], $\Delta^{[V,U]}(\cdot) : \mathbf{F}^{n\times n} \rightarrow \mathbf{F}^{n\times n}$, acting on the linear space $\mathbf{F}^{n\times n}$ which is defined by

$$\Delta^{[V,U]}(C) = VC - CU, \tag{1.1}$$

where $\mathbf{F}$ is a field, $\tilde{s}$ is denoted a vector, C, V, $U \in \mathbf{F}^{n\times n}$ are given matrices. Let the rank r of image $\Delta^{[V,U]}(C)$ smaller than n, then $\Delta^{[V,U]}(C)$ can be re-written non-uniquely as

$$\Delta^{[V,U]}(C) = GH^T, \tag{1.2}$$

where $G, H \in \mathbf{F}^{n\times r}$ are called generator matrices. This displacement structure has been applied to computations of a Cauchy-like matrix, a Toeplit-like matrix and a Hankel-like matrix. We will plan to extend this displacement approach to a new class of structured matrix, referred as a Kronecker-like matrix.

A new class of structured matrix has been formulated based on the displacement structure. As a result, a most efficient algorithm is presented to compute the product of a Kronecker-like matrix over a field using the generators. The memory space cost of the number of the elements stored for a Kronecker-like matrix of the size $n1 \times n1$ over a field is $O(rn)$ with the displacement rank r. The cost of the number of the arithmetic operations for a matrix-vector product is reduced to $O(rn)$.

The organization of this paper is as follows: In the section 2, the definitions and properties of Kronecker matrices are given. In section 3, a Kronecker-like matrix representation of the size $n1 \times n1$ will be presented. In section 4, an efficient algorithm of the multiplication of Kronecker-like matrix will be shown. A summery is included in section 5.

Acknowledgment. I would like to express my deepest appreciation Dr. Pan.

2 Definitions and Properties

In this section, we will going to introduce some definitions and properties. The survey of the displacement structure is summarized as follow.

Definition 2.1 Let $\mathbf{F}$ be denoted as a field. $\mathbf{F}^{n\times n}$ is a linear space. $W \otimes z$ is a tensor space.

Definition 2.2 We denote $\hat{z} = z_0, \cdots, z_{n-1}$ as a vector, where $z_i \in \mathbf{F}$, for $i = 0, \cdots, n-1$.

Definition 2.3 The transpose of a matrix M or a vector $\hat{v}$ over a field $\mathbf{F}$ is written as M^T or $\hat{v}^T$.

Definition 2.4 Let all elements of the vectors $\hat{a}$ and $\hat{b}$ be chosen from a field $\mathbf{F}$. A matrix is referred as a Kronecker matrix, which is defined as

$$S(\hat{a}) \otimes T(\hat{a}) = (\, c_{i,j}\,)_{i,j=0}^{n-1} \qquad (2.1)$$

where $c_{i,j}$ has the form

$$c_{i,j} = \begin{pmatrix} a_0b_0 & a_0b_1 & a_0b_2 & \cdots & a_0b_{n-1} \\ a_1b_0 & a_1b_1 & a_1b_2 & \cdots & a_1b_{n-1} \\ a_2b_0 & a_2b_1 & a_2b_2 & \cdots & a_2b_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1}b_0 & a_{n-1}b_1 & a_{n-1}b_2 & \cdots & a_{n-1}b_{n-1} \end{pmatrix}$$

Proposition 2.1 ([13]) The Kronecker product is a bilinear and associative and satisfies the following: where $L, F, J \in \mathbf{F}^{n\times n}$. and $\alpha \in \mathbf{F}$.

- $L \otimes (F + J) = L \otimes F + L \otimes J$
- $(L + F) \otimes J = L \otimes J + F \otimes J$
- $\alpha(L \otimes F) = L \otimes (\alpha F) = (\alpha L) \otimes F)$
- $(L \otimes F) \otimes J = L \otimes (F \otimes J)$
- $(L \otimes F)(J \otimes Q) = (LJ \otimes FQ)$
- Kronecker product is not commutative.

Lemma 2.1 Let the Sylvester operator, $\Delta^{[P,Q]}(\cdot) : \mathbf{F}^{n\times n} \to \mathbf{F}^{n\times n}$, acting on the linear space $\mathbf{F}^{n\times n}$ which is defined by

$$\Delta^{[P,Q]}(W) = PW - WQ,$$

where P, Q, $W \in \mathbf{F}^{n\times n}$ are given matrices, then the Sylvester operator $\Delta(\cdot)$ is a linear operator.

Proof. Let W_1 and W_2 be any two matrices from $\mathbf{F}^{n\times n}$ and α be an element of $\mathbf{F}$. It is easy to see that $P(W_1 + W_2) - (W_1 + W_2)Q = (PW_1 - W_1Q) + (PW_2 - W_2Q)$ and $P(\alpha W) - (\alpha W)Q = \alpha(PW - WQ)$. Hence, we have the following equations: $\Delta^{[P,Q]}(W_1 + W_2) = \Delta^{[P,Q]}(W_1) + \Delta^{[P,Q]}(W_2)$ and $\Delta^{[P,Q]}(\alpha W) = \alpha\Delta^{[P,Q]}(W)$. *Q.E.D*

Definition 2.5 let x and y are different elements over a field $\mathbf{F}$. The vectors $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$ and $\underline{\mathbf{1}}$ are denoted as $\underline{\mathbf{x}} = x, \cdots, x$, $\underline{\mathbf{y}} = y, \cdots, y$ and $\underline{1} = 1, \cdots, 1$.

Definition 2.6 The inverse of a matrix M over a field $\mathbf{F}$ is written as M^{-1}.

Definition 2.7 A matrix of the size $n1\times n1$ over a field $\mathbf{F}$ is called a Kronecker-like matrix if it associates with matrix $S(\hat{a})\otimes T(\hat{a})$ of (2.1) and satisfies the Sylvester equation of (1.2), where $V = D(\underline{\mathbf{x}})$, $U = D(\underline{\mathbf{y}})$, for the matrices $G, H \in \mathbf{F}^{n\times r}$.

Definition 2.8 The matrices $G, H \in \mathbf{F}^{n\times r}$ and $S(\hat{a}) \otimes T(\hat{a})$ as in the definition 2.7 are called the generators for a Kronecker-like matrix.

3 A Kronecker matrix displacement

In this section, a new representation of a Kronecker-like matrix of the size $n1 \times n1$ over a field has developed. It is easy to see that a Kronecker-matrix of the size $n1 \times n1$ over a field, defined as the definition 2.4 is embedded in a Kronecker-like matrix of the size $n1 \times n1$ over a field. One of the most important property of a Kronecker matrix is having a low displacement rank.

Lemma 3.1 Given an arbitrary vector $\hat{v} \in \mathbf{F}^{n\times 1}$ and a Kronecker-like matrix of the size $n1 \times n1$ over a field as defined as the definition 2.4, then the product $\hat{p} = (S(\hat{a}) \otimes T(\hat{a}))\hat{v}$ can be computed in

$$O(n)$$

operations.

Proof. Based on the proposition 2.1, we have $\hat{p} = (S(\hat{a}) \otimes T(\hat{a}))\hat{v} = S(\hat{a}) \otimes (T(\hat{a})\hat{v})$, the cost of computing the product $\hat{p}$ is $3n-1$ operations. *Q.E.D*

Lemma 3.2 Let $S(\hat{a})\otimes T(\hat{b})$ be a matrix as defined as the definition 2.4. The displacement rank of a $S(\hat{a}) \otimes T(\hat{b})$ matrix is one.

Proof. From on the Proposition 2.1, it follows from the equation below: $\Delta^{[D(\underline{\mathbf{x}}),D(\underline{\mathbf{y}})]}(S(\hat{a})\otimes T(\hat{b})) = D(\underline{x})(S(\hat{a})\otimes T(\hat{b}) - (S(\hat{a})\otimes T(\hat{b}))D(\underline{y}) = D(\underline{x})(S(\hat{a})\otimes T(\hat{b})) - (S(\hat{a})\otimes(D(\underline{y})T(\hat{b})) = (D(\underline{x}) - D(\underline{y}))(S(\hat{a})\otimes T(\hat{b})) = (x-y)GH^T$ where $G = D(\hat{a})\underline{1}$ and $H = \underline{1}D(\hat{b})$. *Q.E.D*

Theorem 3.1 Let the linear operator $\Delta(\cdot)$ act on a tensor space with trivial kernel. Let W be a Kronecker-like matrix of the size $n1 \times n1$ over a field, which defined as definition 2.7 with the generators G, $H \in \mathbf{F}^{n\times r}$ and $S(\hat{a})\otimes T(\hat{b})$ of (2.1). Then this matrix can be represented as

$$W = (\frac{1}{x-y})\sum_{i=1}^{r} S(\hat{g}_i)\otimes T(\hat{h}_i) \qquad (3.1)$$

Proof. Let the matrix W be given as (3.1). From the lemma 2.1, the operator $\Delta(\cdot)$ of (1.1) is a linear operator. By the lemma 3.2, we have $\Delta^{[D(\underline{\mathbf{x}}),D(\underline{\mathbf{y}})]}(W) = \frac{1}{x-y}\Delta^{[D(\underline{\mathbf{x}}),D(\underline{\mathbf{y}})]}$ $(\frac{1}{x-y}\sum_{i=1}^{r} S(\hat{g}_i) \otimes T(\hat{h}_i)) = \frac{1}{x-y}(\sum_{i=1}^{r}\Delta^{[D(\underline{\mathbf{x}}),D(\underline{\mathbf{y}})]}(S(\hat{g}_i)\otimes T(\hat{h}_i))) = \frac{1}{x-y}(D(\underline{x}) - D(\underline{y}))\sum_{i=1}^{r}(S(\hat{g}_i)\otimes T(\hat{h}_i) = \sum_{i=1}^{r} D(\hat{g}_i)D(\hat{\mathbf{1}})D(\hat{\mathbf{1}})^T D(\hat{h}_i) = GH^T$. *Q.E.D*

4 Computation With a Kronecker-like matrix

A superfast algorithm for the multiplication of a Kronecker-like matrix with a vector will be given in this section.

Corollary 4.1 Let $\hat{v} \in \mathbf{F}^{n\times 1}$ be an arbitrary vector over a field $\mathbf{F}$ and Let $W \in \mathbf{F}^{n\times n}$ be a Kronecker-like matrix as the definition 2.7 with displacement rank r. Then the product of $\hat{p} = W\hat{v}$ can be computed in

$$O(rn)$$

operations.

Proof: We rewrite the matrix W as $W = (\frac{1}{x-y})\sum_{i=1}^{r} S(\hat{g}_i)\otimes T(\hat{h}_i)$ as theory 3.1, ie, $\hat{p} = \sum_{i=1}^{r} S(\hat{g}_i)\otimes T(\hat{h}_i)\hat{v} = \sum_{i=1}^{r} S(\hat{g}_i)\otimes(T(\hat{h}_i)\hat{v})$. By the corollary 4.1, we have the complexity $O(rn)$. *Q.E.D*

Algorithm 4.1

Input: an arbitrary vector $\hat{v}$ and a $Kronecker-like$ matrix W of the size $n1 \times n1$ over a field as the definition 2.7 with displacement rank r.

Output: the vector $\hat{p}$ such that $\hat{p} = W\hat{v}$

Computations:

(1) Apply the Corollary 4.1 to compute the vector $\hat{t}_i$ such that $\hat{t}_i = S(\hat{g}_i)\otimes S(\hat{h}_i)\hat{v}$, for $i = 1\cdots, r$.

(2) Sum up the vectors $\hat{t}_i$, $\hat{t} = \sum_{i=1}^{r}\hat{t}_i$.

(3) Compute the vector $\hat{p}$ such that $\hat{p} = (\frac{1}{x-y})\hat{t}$.

5 Conclusion

We have applied the displacement structure to formulated a new class of Kronecker-like matrix of the size $n1\times n1$ over a field. The computational complexity of multiplication with vectors for Kronecker-like matrices has been improved. The displacement approach is not only reduce the running time but also save the memory space in the computer calculation. The number of the elements needed to store in a computer has been saved to $O(rn)$ for a Kronecker-like matrix of the size $n1 \times n1$ over a field. The running time to solve the problem of matrix-vector product has been accelerated to $O(rn)$ arithmetic operations for a class of Kronecker-like matrix with the displacement rank r.

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