Joint Reliability Importance of k-out-of-n Systems and Series-Parallel Systems

Hsun-Wen Chang
Department of Applied Mathematics
Tatung University
Taipei, Taiwan, ROC

Shyr Jan
Department of Applied Mathematics
Tatung University
Taipei, Taiwan, ROC

Abstract - *Joint Reliability Importance (JRI) is a measurement of the degree of interactions between two components in a system. The value of JRI is non-positive (non-negative) if and only if one component becomes more important when the other has failed (is functioning). In this paper, we study the properties of JRI of the k-out-of-n system, where the system works if there are at least k functioning components among n identical and independent distributed components. The variations of JRI according to the probability of the component and to the number of components are analyzed. We also indicate and correct the errors in the previous paper. In addition, the JRI of the series-parallel systems are presented.*

Keywords: Joint reliability importance, *k*-out-of-*n* system, Series-parallel system.

1 Introduction

To measure the contribution of a single component or a specific position in a system towards the system reliability, various reliability importance indices have been proposed and widely studied— Birnbaum importance, combinatorial importance, structural importance, and half-line importance, for example [2-5, 10, 11, 13]. In this paper, we concern the importance index between two components.

Joint reliability importance (JRI) of two components is a measurement of the degree of interactions between two components in a system. The definition of JRI is as follows.

$$JRI\,(i,j) = E[\phi(1_i,1_j,X)] + E[\phi(0_i,0_j,X)] - E[\phi(1_i,0_j,X)] - E[\phi(0_i,1_j,X)], \tag{1}$$

where

$$\phi(\cdot_i,\cdot_j,X) = \phi(X_1,\cdots,X_{i-1},\cdot_i,X_{i+1},\cdots,X_{j-i},\cdot_j,X_{j+1},\cdots,X_n) \tag{2}$$

denotes the states of n components. JRI was first proposed independently by Hagstrom [6] and by Hong and Lie [9], and then attracted many researches [1, 7-9, 12]. The value of JRI is non-positive (non-negative) if and only if one component becomes more important when the other has failed (is functioning). Hong et al. proposed several properties of JRI of the k-out-of-n system [8]. However, some of their results are incorrect.

In this paper, we re-state the properties of JRI of the 2-out-of-n system and correct the errors in the previous paper. We also compare the JRI of the k-out-of-n system and that of the k-out-of-$(n+1)$ system. In addition, we present the JRI of the series-parallel system.

2 Joint Reliability Importance of k-out-of-n Systems

In this section, we study the joint reliability importance of the k-out-of-n system with identical and independent distributed (IID) components. A k-out-of-n system is a system that works if there are at least k functioning components among n components. Hong et al. stated the definition of JRI for the IID model. They derived the JRI and then characterized the sign of JRI of the k-out-of-n system with IID components as follows [8].

Definition 2.1. *The JRI for the IID model is*

$$JRI(i,j) = R(1_i,1_j,p) + R(0_i,0_j,p) - R(1_i,0_j,p) - R(0_i,1_j,p), \tag{3}$$

where $R(\bullet)$ is the system reliability and p denotes the probability of a component to be functioning.

Theorem 2.2. (See [8]) *For $n \ge 3$ and $2 \le k \le n$, the JRI of the k-out-of-n system with IID components is*

$$JRI_{k,n} = p^{k-2}(1-p)^{n-k-1}\left[\binom{n-2}{k-2} - \binom{n-1}{k-1}p\right] \tag{4}$$

for any two components i and j of the system.

Corollary 2.3. (See [8]) *Consider the k-out-of-n system with IID components for $n \ge 3$ and $2 \le k \le n$.*

(i) $JRI_{k,n} > 0$ *if* $0 < p < \frac{k-1}{n-1}$.

(ii) $JRI_{k,n} = 0$ *if* $p = \frac{k-1}{n-1}$.

(iii) $JRI_{k,n} < 0$ *if* $\frac{k-1}{n-1} < p < 1$.

In Corollary 2.3, it means that the JRI of two components is positively related for a smaller probability p and negatively related for a larger p. Moreover, the threshold is $p = \frac{k-1}{n-1}$.

In the following, we consider the properties of JRI of 2-out-of-n systems, denoted by $JRI_{2,n}(p)$, for $n \ge 3$.

Theorem 2.4. *For the 2-out-of-n system,*

(i) $JRI_{2,n}(p) = (1-p)^{n-3}[1-(n-1)p]$.

(ii) $\lim_{p \to 0} JRI_{2,n}(p) = 1$ *and* $\lim_{p \to 1} JRI_{2,n}(p) = 0$.

(iii) *The minimum of* $JRI_{2,n}(p)$ *is* $-\left(\frac{n-3}{n-1}\right)^{n-3}$ *when* $p = \frac{2}{n-1}$.

(iv) *For* $n \ge 5$, *the graph of* $JRI_{2,n}(p)$ *has a point of inflection* $\left(\frac{3}{n-1}, -2\left(\frac{n-4}{n-1}\right)^{n-3}\right)$.

Proof. Statement (i) follows Theorem 2.2 by $k = 2$ and thus we obtain Statement (ii). Statements (iii) and (iv) can be obtained by the first-order derivative and the second-order of derivative of JRI with respect to p. □

Figure 1 shows the graphs of $JRI_{2,n}(p)$ of 2-out-of-n systems with IID components for $3 \le n \le 8$. Due to Theorem 2.4, each graph of JRI for $n \ge 5$ has a point of inflection and thus is not convex, which is misstated convex by Hong et al. [8]. Note that the minimum value of JRI indicates the maximum negatively related, which occurs as $p = \frac{2}{n-1}$. When $p \ge \frac{2}{n-1}$, the JRI increases as p increases and the increment decreases as $p \ge \frac{3}{n-1}$.

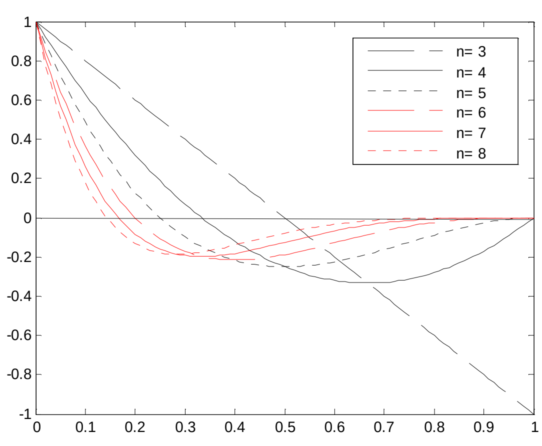

Figure 1. The JRI of 2-out-of-n systems for $3 \le n \le 8$.

In the following, we compare the JRIs of 2-out-of-n system $JRI_{2,n}(p)$ and 2-out-of-$(n+1)$ system $JRI_{2,n+1}(p)$.

Theorem 2.5.

(i) $JRI_{2,n}(p) > JRI_{2,n+1}(p)$ *if* $0 < p < \frac{2}{n}$.

(ii) $JRI_{2,n}(p) = JRI_{2,n+1}(p)$ *if* $p = \frac{2}{n}$.

(iii) $JRI_{2,n}(p) < JRI_{2,n+1}(p)$ *if* $\frac{2}{n} < p < 1$.

Proof. By Theorem 2.4 (i),

$$JRI_{2,n}(p) - JRI_{2,n+1}(p) = p(1-p)^{n-3}(2-np). \quad □$$

Note that, when $p = \frac{2}{n}$, the relationship between two components in the 2-out-of-n system and the 2-out-of-$(n+1)$ system is the same. When $\frac{2}{n} < p < 1$, the relationship between two components is more negatively related in the 2-out-of-n system than in the 2-out-of-$(n+1)$ systems. Hong et al. also compared the JRIs of 2-out-of-n system and 2-out-of-$(n+1)$ system [8]. However, they misstated $JRI_{2,n}(p) < JRI_{2,n+1}(p)$ when $p > 1/2$.

In the following, we consider the joint reliability importance for general k. We also make comparison of JRIs for k-out-of-n system and k-out-of-$(n+1)$ system.

Theorem 2.6.

(i) *For* $3 \le k \le n-2$, $\lim_{p \to 0} JRI_{k,n}(p) = \lim_{p \to 1} JRI_{k,n}(p) = 0$.

(ii) $\lim_{p \to 0} JRI_{n-1,n}(p) = 0$ *and* $\lim_{p \to 1} JRI_{n-1,n}(p) = -1$.

Proof. By Theorem 2.2. □

Theorem 2.7. *Let* $a = \sqrt{\frac{(k-1)(n-k+1)}{n-1}}$. *For* $3 \le k \le n-1$, *the comparison of JRIs between the k-out-of-n system and the k-out-of-$(n+1)$ system is as follows.*

(i) $JRI_{k,n}(p) < JRI_{k,n+1}(p)$ *if* $0 < p < \frac{k-1-a}{n}$ *or* $\frac{k-1+a}{n} < p < 1$.

(ii) $JRI_{k,n}(p) = JRI_{k,n+1}(p)$ *if* $p = \frac{k-1 \pm a}{n}$.

(iii) $JRI_{k,n}(p) > JRI_{k,n+1}(p)$ *if* $\frac{k-1-a}{n} < p < \frac{k-1+a}{n}$.

Proof. Similar to the proof of Theorem 2.5. □

Note that we completely solve the comparison of JRIs between the k-out-of-n system and the k-out-of-$(n+1)$ system. Roughly, $JRI_{k,n}(p) < JRI_{k,n+1}(p)$ if

$$\left| p - \frac{k-1}{n} \right| > \frac{\sqrt{k-1}}{n}. \tag{5}$$

Figure 2 shows the graphs of $JRI_{3,n}(p)$ for $4 \le n \le 8$.

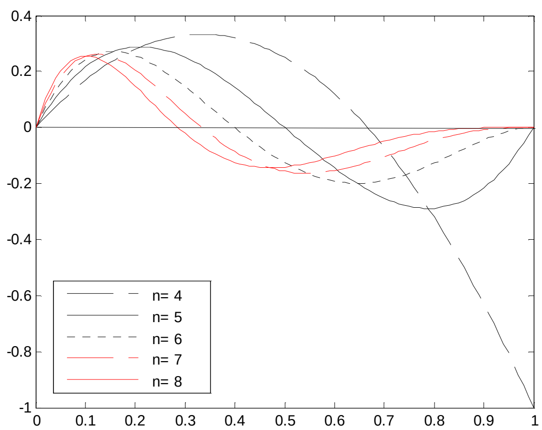

Figure 2. The JRIs of 3-out-of-n systems for $4 \le n \le 8$.

3 Joint Reliability Importance of Series-Parallel Systems

In this section, we study the joint reliability importance of the series-parallel system with identical and independent distributed (IID) components. Let the series-parallel system S have k subsystems connected in parallel where each subsystem has n_i IID components connected in series. See Figure 3 for example. In the following, we present the joint reliability importance (JRI) of the series-parallel system.

Theorem 3.1. *The reliability of the series-parallel system is* $R = 1 - \prod_i (1 - p^{n_i})$.

Proof. Note that a series subsystem S_i is functioning if all

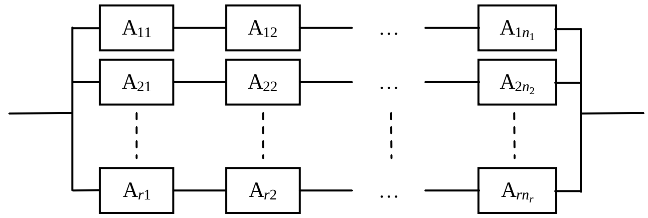

Figure 3. A series-parallel system.

components of it are functioning. Thus the reliability of S_i is p^{n_i}. The system S is failed if and only if all subsystems are failed. Hence Theorem 3.1 can be obtained. □

Theorem 3.2. *The joint reliability importance of the series-parallel system is as follows.*

(i) $JRI(A_{ju}, A_{jv}) = p^{n_j - 2} \prod_{i \ne j} (1 - p^{n_i})$.

(ii) $JRI(A_{ju}, A_{kv}) = -p^{n_j + n_k - 2} \prod_{i \ne j,k} (1 - p^{n_i})$ *for* $j \ne k$.

Proof. By Definition 2.1 and Theorem 3.1, for any two components in the same subsystem, we have the reliabilities

$$R(1_{A_{ju}}, 1_{A_{jv}}, p) = 1 - (1 - p^{n_j - 2}) \prod_{i \ne j} (1 - p^{n_i}),$$

$$R(0_{A_{ju}}, 0_{A_{jv}}, p) = R(1_{A_{ju}}, 0_{A_{jv}}, p)$$
$$= R(0_{A_{ju}}, 1_{A_{jv}}, p) = 1 - \prod_{i \ne j} (1 - p^{n_i}).$$

Hence $JRI(A_{ju}, A_{jv}) = p^{n_j - 2} \prod_{i \ne j} (1 - p^{n_i})$.

Similarly, for two components in distinct series subsystems, we have

$$R(1_{A_{ju}}, 1_{A_{kv}}, p) = 1 - (1 - p^{n_j - 1})(1 - p^{n_k - 1}) \prod_{i \ne j,k} (1 - p^{n_i}),$$

$$R(0_{A_{ju}}, 0_{A_{kv}}, p) = 1 - \prod_{i \ne j,k} (1 - p^{n_i}),$$

$$R(1_{A_{ju}}, 0_{A_{kv}}, p) = 1 - (1 - p^{n_j - 1}) \prod_{i \ne j,k} (1 - p^{n_i}),$$

$$R(0_{A_{ju}}, 1_{A_{kv}}, p) = 1 - (1 - p^{n_k - 1}) \prod_{i \ne j,k} (1 - p^{n_i}).$$

Hence $JRI(A_{ju}, A_{kv}) = -p^{n_j + n_k - 2} \prod_{i \ne j,k} (1 - p^{n_i})$. □

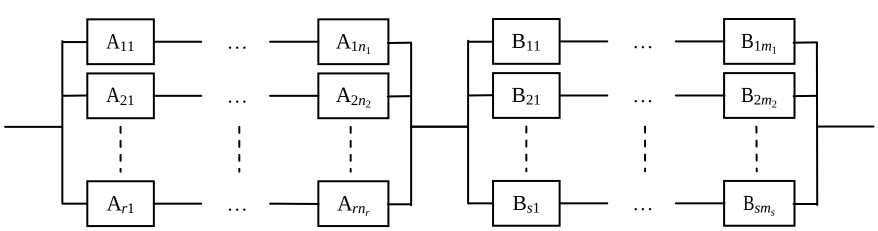

Figure 4. A general series-parallel system.

Note that the JRI of two components in the same series subsystem is positive, while the JRI of those in distinct subsystem is negative. Similarly, the JRI of two components in a general series-parallel system— Figure 4, for example, can be obtained.

4 Conclusion

In this paper, we study the joint reliability importance (JRI) of the k-out-of-n system and the series-parallel system with IID components. We present the properties of JRI of the k-out-of-n system and point out some errors in literature. For $k=2$, the JRI has a minimum as $p=2/(n-1)$. For $k=2$ and $n \geq 5$, the JRI has a point of inflection as $p=3/(n-1)$. We make exact comparison of the JRIs between k-out-of-n system and k-out-of-$(n+1)$ system. In the 2-out-of-n system, for a small p, the JRI of the 2-out-of-n system is larger than that of 2-out-of-$(n+1)$ system. But in the k-out-of-n system, for a small p or a large p, the JRI of the k-out-of-n system is less than that of k-out-of-$(n+1)$ system.

In addition, we present the JRI of the series-parallel system. The result can be easily extended to the general series-parallel systems.

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