Average Distances of Pyramid Networks

Hsien-Jone Hsieh and Dyi-Rong Duh
National Chi Nan University
No. 1, University Rd., Puli, Nantou Hsien, Taiwan 54561, R.O.C.
Phone: 886 49 2910960#4824 Fax: 886 49 2915226
{s1321902,drduh}@ncnu.edu.tw
Correspondence to: D.-R. Duh; e-mail: drduh@ncnu.edu.tw

Abstract

For an interconnection network, calculating average distance of it is in general more difficult than determining its diameter. Diameters of pyramid networks are well known. This study calculates average distances of pyramid networks.

Keywords: Interconnection networks, pyramid networks, average distance, shortest-path, routing algorithm

1. Introduction

An interconnection network can be represented as an undirected graph where a processor is represented as a *node*, and a communication channel between processors as an *edge* between corresponding nodes [21]. Besides, those well-known structures such as mesh, tree, hypercube and its various cousins, the pyramid structure has long been proposed for parallel computing, network computing, and image processing [1], [4], [5], [9], [11], [14], [18], [19], [21]. Some topological properties of the pyramid network such as diameter, shortest routing, connectivity, Hamiltonicity and pancyclicity were investigated in [1], [3], [8], [14], [20], [21].

In most applications, communication is undoubtedly the most crucial issue for massively parallel computers due to it takes much more time than computation. As a basic support for running any parallel algorithm in distributed systems, an efficient routing scheme should be provided for them to exchange information between processors in the interconnected network structure [10], [12], [13]. The *routing problem* consists of finding a path through which a message can be transmitted from the source node to the destination node in a network. A *routing algorithm* is a solution to the routing problem.

Pyramid networks, will be formally defined in Section 2, is a hierarchy structure and immediately leads to a simple routing scheme, called the *UP-DOWN routing* [6]. Some single path and multiple paths routing algorithms between any two nodes in pyramid networks were provided in [2], [7], [15], [16], [17]. Hsieh et al. demonstrate a shortest path routing algorithm for pyramid networks in 2004 [8]. They first provide a simple routing algorithm for pyramid networks by modifying the UP-DOWN routing. In worst case, the algorithm takes $O(n)$ time to construct a path between any two distinct nodes in a n-layer pyramid network. Besides, it cannot always generate a shortest path. Naturally, the length of a path between two arbitrary nodes in a network bounds below the time complexity of outputting the path. Then they provide a shortest path routing algorithm which takes $O(1)$ time to determine the structure and compute the length of a shortest path between any two distinct nodes in a pyramid network. Based on their idea, the distance between any two nodes in a pyramid network can be determined. This work calculates average distances of pyramid networks by a recursive manner.

The remainder of this paper is organized as follows. In the next section, pyramid networks are formally described. Then some notations used in this paper are also defined. Section 3 reveals how to determine a shortest path in a pyramid network. Section 4 states how to determine the average distance of a pyramid network. Finally, we summarize the results in Section 5.

2. Preliminaries

This section describes the pyramid structure formally. For convenience and simplicity, some notations are also defined later.

A pyramid network (*PM*, for short) is a hierarchical structure and combined by a 4-ary tree and meshes. A mesh, $M(m, n)$, is a set of nodes $V(M(m, n)) = \{(x, y) \mid 0 \le x < m, 0 \le y < n\}$ and two nodes, (x_1, y_1) and (x_2, y_2), are connected by an edge iff $|x_1 - x_2| + |y_1 - y_2| = 1$. A n-layer *PM*, denoted by $PM[n]$, is a set of nodes $V(PM[n]) = \{(l; x, y) \mid 0 \le l \le n, 0 \le x, y < 2^l\}$, where $n \ge 1$. A node, $(l; x, y) \in V(PM[n])$, is said to be a node at layer l. x (y) can be represented by a decimal or an l-bit binary number. An l-bit binary number x (y) can be denoted by $x(1..l)$ ($y(1..l)$) where bit 1 is the most significant bit and bit l is the least most significant bit. The 4^l nodes at layer $l \le n$ are connected as a $M(2^l, 2^l)$ and can be divided into four $M(2^{l-1}, 2^{l-1})$s with 4^{l-1} nodes. Let l_u, x_u and y_u denote the layer, x coordinate, and y coordinate of a node $u=(l_u; x_u, y_u)$. The node $(l; x, y)$ is also connected to $(l-1; \lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$, $(l+1; 2x, 2y)$, $(l+1; 2x, 2y+1)$, $(l+1; 2x+1, 2y)$, and $(l+1; 2x+1, 2y+1)$ for $0<l<n$. The *apex* of $PM[n]$ (apex, for short) is defined to be (0; 0, 0) and only adjacent to (1; 0, 0), (1; 0, 1), (1; 1, 0), and (1; 1, 1), therefore the order of $PM[n]$ is $\frac{4^{n+1}-1}{3}$. That is the degree of the apex is 4. The edges contained in a mesh of $PM[n]$ are called *mesh-edges*, and those edges connecting two consecutive layers are called *layer-edges*. Figure 1 and Figure 2 show the literal view and top view of $PM[2]$, respectively, and some nodes in both figures are identified.

For a node $v = (l; x, y)$, the node $P(v)= (l-1; \lfloor x/2 \rfloor, \lfloor y/2 \rfloor)$ is its *parent*. Conversely, v is a *child* of $P(v)$ and is denoted by $C(v)$. Obviously, the node $P(v)$ has four *children*. Let $P^i(v)$ ($C^i(v)$) denote the i-th ancestor (descendant) of a node v. $P^i(v)$ ($C^i(v)$) can be defined as follows:

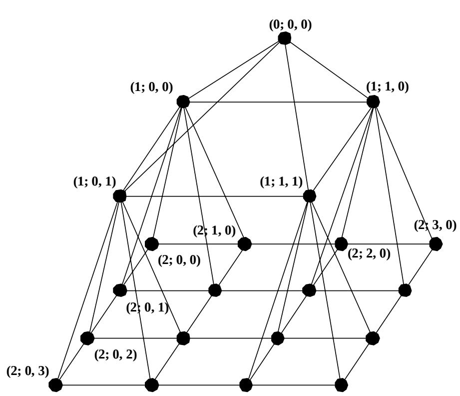


Figure 1. The lateral view of $PM[2]$.

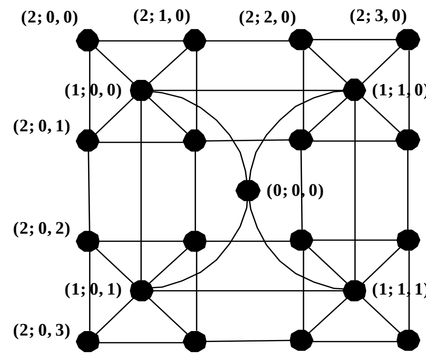


Figure 2. The top view of $PM[2]$.

1. $i=0$, $P^0(v)$ ($C^0(v)$) $= v$.
2. $i=1$, $P^1(v) = P(v)$ ($C^1(v) = C(v)$) is simply the parent (a child) of v.
3. $i \ge 1$, $P^i(v) = (l-i; x(1..l-i), y(1..l-i))$ ($C^i(v)$ at layer $l+i \le n$) is the parent (a child) of $P^{i-1}(v)$ ($C^{i-1}(v)$).

For consistency, in the rest of this paper, let $u=(l_u; x_u, y_u)$ and $v=(l_v; x_v, y_v)$ be two distinct nodes in $PM[n]$, and without loss of generality, assume $l_u \ge l_v$. Given two paths P_1 and P_2, let $P_1 \rightarrow P_2$ denote joining P_2 to the tail of P_1. Three types of paths from u to v in $PM[n]$ are defined as $P_M(u, v)$, $P_A(u, v)$, and $P_V(u, v)$. $P_M(u, v)$ is a shortest path from u to v in the $M(2^l, 2^l)$ of $PM[n]$ when $l_u=l_v$. Then $P_M(u, v)$ can be constructed as $u=(l_u; x_u, y_u) \rightarrow \dots \rightarrow (l_u; x_v, y_u) \rightarrow \dots \rightarrow (l_u; x_v, y_v)=v$ or $u=(l_u; x_u, y_u) \rightarrow \dots \rightarrow (l_u; x_u, y_v) \rightarrow \dots \rightarrow (l_u; x_v, y_v)=v$, and the length of $P_M(u, v)$ can be easily computed as $|x_u-x_v| + |y_u-y_v|$. Note that a $P_M(u, v)$ only contains mesh-edges. $P_A(u, v)$ ($P_V(u, v)$) is a path uses layer-edges to climb up (down) i layers to node u', then connected to a $P_M(u', v')$, finally uses layer-edges again to climb down (up) from v' to v, where $u'=P^i(u)$, $v'=P^j(v)$ ($u'=C^i(u)$, $v'=C^j(v)$) and $j \le i \le l_u$ ($l_v+j= l_u+i \le n$). The lengths of $P_M(u, v)$, $P_A(u, v)$, and $P_V(u, v)$ are denoted by $d_M(u, v)$, $d_A(u, v)$ and $d_V(u, v)$, respectively. Let $d_M^i(u, v)$ denote the length of $P_M(P^i(u), P^i(v))$ for $i \ge 1$. It is clear that $d_M^i(u, v)$s are all shortest.

3. Shortest path

Before constructing a shortest path in a pyramid network, some notations should be defined first. Let $P(u, v)$ ($P^*(u, v)$) denote a path (a shortest path) between two nodes u and v in $PM[n]$ and $d(u, v)$ ($d^*(u, v)$) represent its length. In the following, we first show that one of $P_A(u, v)$s is a shortest path first and then determine the structure of a $P^*(u, v)$.

Lemma 1 [8]. $P_V(u, v)$ is not a $P^*(u, v)$ if $l_u=l_v$.

Lemma 1 shows that $P_V(u, v)$ is not contained in a shortest path and can be replaced by a shorter path which is $P_M(u, v)$.

Lemma 2 [8]. $P_M(u, v) \rightarrow P^*(v, P^i(v))$ is not shorter than $P^*(u, P^i(u)) \rightarrow P_M(P^i(u), P^i(v))$ if $l_u = l_v$ and $i \leq l_u$.

Lemma 2 means that $P_M(u, v) \rightarrow P^*(v, P^i(v))$ can be replaced by $P^*(u, P^i(u)) \rightarrow P_M(P^i(u), P^i(v))$ without increasing length. By Lemmas 1 and 2, we can conclude that one of $P_A(u, v)$s is shortest.

Lemma 3 [8]. If $d_M(u, v) < 2 + d_M^1(u, v)$ then $d_M(u, v) \leq 4$, where $l_u = l_v$.

Lemma 3 shows that $d_M(u, v) \geq d^*(u, v)$ if $d_M(u, v) \geq 5$. In other words, $P_M(u, v)$ may not be a shortest path between nodes u and v in $PM[n]$ when $d_M(u, v) \geq 5$.

Lemma 4 [8]. If $d_M(u, v) < d_M^1(u, v)+2$, then $d_M(u, v) \leq d_A(u, v)$ if $l_u = l_v$.

Lemma 4 tells us that if $d_M(u, v) < d_M^1(u, v)+2$ then $P_M(u, v)$ is $P^*(u, v)$. In the following, we assume that u' (v') be an ancestor of u (v). Let $P_M(u', v')$ be the path contained in $P_A(u, v)$, we now show that the $P_A(u, v)$ is a shortest path if $d_M(u', v') < d_M^1(u', v') + 2$.

Theorem 5 [8]. The $P_A(u, v)$ containing $P_M(u', v')$ is shortest if $d_M(u', v') < d_M^1(u', v')+2$.

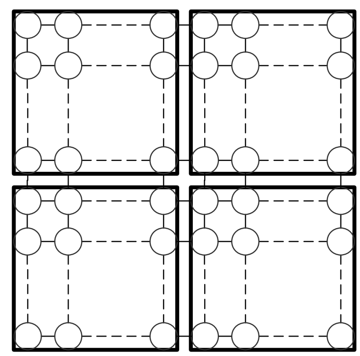
4. Average Distance

A graph is connected if every pair of nodes in it joined by a path. The *average distance* of a connected graph G, denoted by $AD(G)$, is the average over all distances $d(u, v)$ for nodes u and v in G. Hence, we have $AD(G) = \frac{\sum_{u \in G} \sum_{v \in G} d(u, v)}{V(G) \times (V(G)-1)}$, where $\sum_{u \in G} \sum_{v \in G} d(u, v)$ is the summation of the distances between all pairs of nodes in G and $V(G) \times (V(G)-1)$ is number of node pairs. Based on this definition and Theorem 5, the average distance of $PM[n]$ can be now determined. For a $PM[n]$, let $TD(n)$ denote the summation of the distances between all pairs of nodes. Since the order of $PM[n]$ is $\frac{4^{n+1}-1}{3}$, $AD(PM[n]) = \frac{9 \times TD(n)}{16^{n+1} - 5 \times 4^{n+1} + 4}$. Let u and v be two nodes at layer j and i in $PM[n]$, respectively, without loss of generality, assume that $j \geq i$. Since $d(u, v) = (j-i) + d(u', v)$ where $u' = P^{j-i}(u)$, we have $TD(n) = \sum_{u \in PM[n]} \sum_{v \in PM[n]} d(u, v) = \sum_{0 \leq i \leq j \leq n} \sum_{u \in M(2^j, 2^j)} \sum_{v \in M(2^i, 2^i)} (d(u, v)+d(v, u)) - \sum_{0 \leq i \leq n} \sum_{u' \in M(2^i, 2^i)} \sum_{v \in M(2^i, 2^i)} (d(u, v)+d(v, u))$. Because u' has 4^{j-i} descendants at j-layer mesh, the distances from all of them to v are $(j-i) + d(u', v)$. Since there are 4^i nodes and 4^j nodes at layer i and layer j respectively, we have $TD(n) = \sum_{0 \leq i \leq j \leq n} (2 \times (j-i) \times 4^{i+j} + 2 \times 4^{j-i} \times \sum_{u' \in M(2^i, 2^i)} \sum_{v \in M(2^i, 2^i)} (d(u', v)+d(v, u')) - \sum_{0 \leq i \leq n} \sum_{u' \in M(2^i, 2^i)} \sum_{v \in M(2^i, 2^i)} (d(u', v)+d(v, u'))$. The terms in $TD(n)$ excluding the term $\sum_{u' \in M(2^i, 2^i)} \sum_{v \in M(2^i, 2^i)} (d(u', v)+d(v, u'))$ can be now calculated. Let $TD_M(i)$ denote $\sum_{u' \in M(2^i, 2^i)} \sum_{v \in M(2^i, 2^i)} (d(u', v)+d(v, u'))$ and $TD(n)$ can be reformulated as shown in Equation (1).

$$TD(n) = \frac{8}{135} \times 16^{n+1} - (\frac{2}{9} \times n + \frac{2}{9}) \times 4^{n+1} - \frac{8}{135} + 2 \times \sum_{0 \leq j \leq n} (4^j \times \sum_{0 \leq i \leq j} \frac{TD_M(i)}{4^i}) - \sum_{0 \leq i \leq n} TD_M(i) \qquad (1)$$

Now our mission is turn to determine $TD_M(h)$, where $0 \leq h = i \leq n$. In Figure 3, a $M(2^h, 2^h)$ can be divided into four $M(2^{h-1}, 2^{h-1})$s H, I, J, and K. Three cases for determining $TD_M(h)$ should be discussed as follows:

H:$M(2^{h-1}, 2^{h-1})$ I:$M(2^{h-1}, 2^{h-1})$



J:$M(2^{h-1}, 2^{h-1})$ K:$M(2^{h-1}, 2^{h-1})$

Figure 3. Four $M(2^{h-1}, 2^{h-1})$s in a $M(2^h, 2^h)$.

Case 1. (both u and v in the same $M(2^{h-1}, 2^{h-1})$): There are four such $M(2^{h-1}, 2^{h-1})$s and each $TD_M(h-1) = \sum_{u \in M(2^{h-1}, 2^{h-1})} \sum_{v \in M(2^{h-1}, 2^{h-1})} (d(u, v)+d(v, u))$. Notably, $TD_M(h-1)$ can be recursively computed.

Case 2. (u and v are in two adjacent $M(2^{h-1}, 2^{h-1})$): Let these two $M(2^{h-1}, 2^{h-1})$s are H and I, there are four such cases and the summation of the distances of all pairs of nodes in two adjacent $M(2^{h-1}, 2^{h-1})$s, denoted by $TD_A(h-1)$, is equal to $\sum_{u \in H} \sum_{v \in I} (d(u, v)+d(v, u)) = 2 \sum_{u \in H} \sum_{v \in I} d(u, v)$.

Case 3. (u and v are in two diagonal $M(2^{h-1}, 2^{h-1})$s): Let these two $M(2^{h-1}, 2^{h-1})$s are H and K, there are two such cases, the summation of the distances of all pairs of nodes in two diagonal meshes, denoted by $TD_D(h-1)$, is equal to $\sum_{u \in H} \sum_{v \in K} (d(u, v)+d(v, u)) = 2 \sum_{u \in H} \sum_{v \in K} d(u, v)$.

According to the cases described above, we have

$$TD_M(0) = 0,\ TD_M(1) = 16,\ TD_M(2) = 600,\ TD_M(3) = 16728 \qquad (2a)$$

$TD_M(h)=4\times TD_M(h-1)+4\times TD_A(h-1)+2\times TD_D(h-1)$, for $h\geq 4$ (2b)

Equation (2a) shows the value of $TD_M(h)$ where $0\leq h\leq 3$. In the right hand side of Equation (2b), each product term is contributed by a case mentioned above.

Let $TD_A(h)$ denote the summation of the distances of all pairs of nodes in two adjacent $M(2^h, 2^h)$s H and I in a $M(2^{h+1}, 2^{h+1})$ as shown in Figure 4. The meshes H and I are divided into eight $M(2^{h-1}, 2^{h-1})$s named H_1, H_2, H_3, H_4, I_1, I_2, I_3, and I_4. Then, $TD_A(h)=2\sum_{u\in H}\sum_{v\in I} d(u, v) = 2\sum_{u\in\{H_1 \text{ or } H_2 \text{ or } H_3 \text{ or } H_4\}}\sum_{v\in\{I_1 \text{ or } I_2 \text{ or } I_3 \text{ or } I_4\}} d(u, v)$. Fives cases should be discussed as follows:

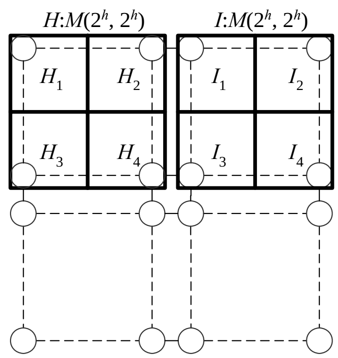


Figure 4. Two adjacent $M(2^h, 2^h)$s H and I in $M(2^{h+1},2^{h+1})$.

Case1. (u in H_2 (H_4) and v in I_1 (I_3)): $TD_A(h-1) = 2\sum_{u\in H_2}\sum_{v\in I_1} d(u, v) = 2\sum_{u\in H_4}\sum_{v\in I_3} d(u, v)$. Note that $TD_A(h-1)$ can be recursively calculated.

Case 2. (u in H_2 (H_4) and v in I_3 (I_1)): $TD_D(h-1) = 2\sum_{u\in H_2}\sum_{v\in I_3} d(u, v) = 2\sum_{u\in H_4}\sum_{v\in I_1} d(u, v)$.

Case 3. (u in H_r and v in I_r, where $1\leq r\leq 4$): As shown in Figure 5, H_r and I_r can be divided into eight $M(2^{h-2}, 2^{h-2})$s named H_{r1}, H_{r2}, H_{r3}, H_{r4}, I_{r1}, I_{r2}, I_{r3}, and I_{r4}. The shortest path from u in H_{r2} (H_{r4}) to v in I_{r1} (I_{r3}) is $P^*(u, P^{h-2}(u))\rightarrow P^*(P^{h-2}(u), P^{h-2}(v))\rightarrow P^*(P^{h-2}(v), v)$. Other than these two cases, the shortest path from u to v is $P^*(u, P^{h-1}(u))\rightarrow P^*(P^{h-1}(u), P^{h-1}(v))\rightarrow P^*(P^{h-1}(v), v)$. Note that $d_M(P^{h-2}(u), P^{h-2}(v))=3$ and $d_M(P^{h-1}(u), P^{h-1}(v))=2$. Thus, $2\sum_{u\in H_r}\sum_{v\in I_r} d(u, v) = 2\times(2\times(2\times(h-2)+3)\times(4^{h-2})^2 + 14\times(2\times(h-1)+2)\times(4^{h-2})^2)$.

Case 4. (u in H_r and v in I_s, where $|r-s|=2$ and $1\leq r, s\leq 4$): This case is similar to Case 3 and it is easy to check that $2\sum_{u\in H_r}\sum_{v\in I_s} d(u, v) = 2\times((2\times(h-2)+4)\times(4^{h-2})^2 + 15\times(2\times(h-1)+3)\times(4^{h-2})^2))$.

Case 5. (u in H_r and v in I_s, where r = 1 or 3 and s = 2 or 4): $P^*(u, v) = P^*(u, P^h(u))\rightarrow P_M(P^h(u), P^h(v))\rightarrow P^*(P^h(v), v)$ and $d_M(P^h(u), P^h(v))=1$. Therefore, $2\sum_{u\in H_r}\sum_{v\in I_s} d(u, v)$ is equal to $2\times((2\times h + 1)\times(4^{h-1})^2)$.

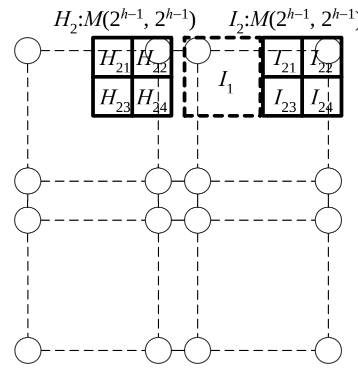


Figure 5. The partitions of 2 $M(2^{h-1}, 2^{h-1})$s H_2 and I_2.

According to the cases described above, we have

$TD_A(0)=2$, $TD_A(1)=76$, $TD_A(2)=2152$ (3a)

$TD_A(h)=2\times TD_A(h-1)+2\times TD_D(h-1)+(\frac{3}{2}\times h+\frac{29}{64})\times 16^h$, for $h\geq 3$ (3b)

Equation (3a) shows the value of $TD_A(h)$ where $0\leq h\leq 2$. In the right hand side of Equation (3b), the first (second) term is contributed by Case 1 (Case 2) and third term is contributed by Cases 3, 4, and 5.

Let $TD_D(h)$ denote the summation of the distances of all pairs of nodes in two diagonal $M(2^h, 2^h)$s H and K in a $M(2^{h+1}, 2^{h+1})$ as shown in Figure 6. The meshes H and K are divided into eight $M(2^{h-1}, 2^{h-1})$s named H_1, H_2, H_3, H_4, K_1, K_2, K_3, and K_4. Then, $TD_D(h)=2\sum_{u\in H}\sum_{v\in K} d(u, v) = 2\sum_{u\in\{H_1 \text{ or } H_2 \text{ or } H_3 \text{ or } H_4\}}\sum_{v\in\{K_1 \text{ or } K_2 \text{ or } K_3 \text{ or } K_4\}} d(u, v)$. Three cases should be discussed as follows:

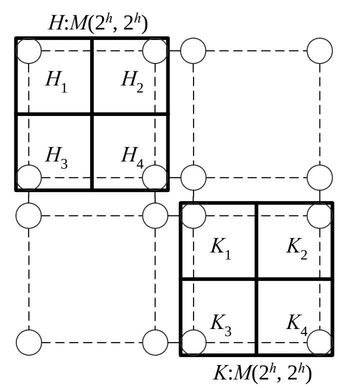


Figure 6. Two diagonal $M(2^h, 2^h)$s H and K in a $M(2^{h+1}, 2^{h+1})$.

Case 1. (u in H_4 and v in K_1): $TD_D(h-1) = 2\sum_{u\in H_4}\sum_{v\in K_1} d(u, v)$.

Case 2. (u in H_r and v in K_r, $1\leq r\leq 4$): $P^*(u, v) = P^*(u, P^{h+1}(u))\rightarrow P^*(P^{h+1}(v), v)$. Note that $P^{h+1}(u) = P^{h+1}(v)$. $2\sum_{u\in H_r}\sum_{v\in K_r} d(u, v) = 2\times(2\times(2\times(h+1))\times(4^{h-1})^2)$.

Case 3. (u in H_4 and v in K_r, r is 2 or 3): H_4 and K_r are divided into eight $M(2^{h-2}, 2^{h-2})$s named H_{41}, H_{42}, H_{43}, H_{44}, K_{r1}, K_{r2}, K_{r3}, and K_{r4} shown in Figure 7. The shortest path from u in H_{44} to v in K_{r1} is $P^*(u, P^{h-2}(u))\rightarrow P^*(P^{h-2}(u), P^{h-2}(v))\rightarrow P^*(P^{h-2}(u), v)$. Other than this case, the shortest path from u to v is $P^*(u, P^{h-1}(u))\rightarrow P^*(P^{h-1}(u), P^{h-1}(v))\rightarrow P^*(P^{h-1}(v), v)$. Note that $d_M(P^{h-2}(u), P^{h-2}(v))=4$ and $d_M(P^{h-1}(u), P^{h-1}(v))=3$. Thus, $2\sum_{u\in H_4}\sum_{v\in K_2} d(u, v) = 2\times((2\times(h-2)+4)\times(4^{h-2})^2 + 15\times(2\times(h-1)+3)\times(4^{h-2})^2)$.

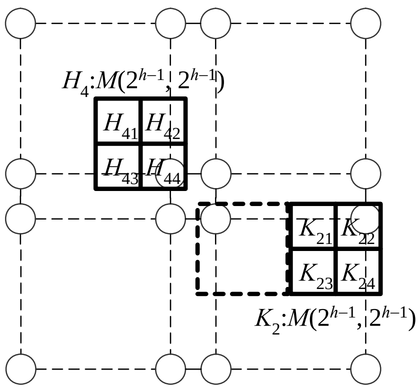


Figure 7. The partitions of two $M(2^{h-1}, 2^{h-1})$s H_4 and K_2.

Case 4. (u in H_r and v in K_1, r is 2 or 3): This case is similar to Case 3 and it is easy to check that $2\sum_{u\in H_4}\sum_{v\in K_2} d(u, v) = 2\times((2\times(h-2)+4)\times(4^{h-2})^2 + 15\times(2\times(h-1)+3)\times(4^{h-2})^2)$.

According to the cases described above, we have

$TD_D(0)=4$, $TD_D(1)=116$ (4a)

$TD_D(h)= TD_D(h-1) + (\frac{15}{8}\times h+\frac{103}{64})\times 16^h = (4\times h+\frac{19}{6})\times 16^h + \frac{4}{3}$, for $h\geq 2$. (4b)

Equation (4a) shows the value of $TD_D(h)$ where $0\leq h\leq 1$. In the right hand side of Equation (3b), the first term is contributed by Case 1 and second term is contributed by Cases 2, 3, and 4.

By Equations (4a) and (4b), Equation (3b) can be rewritten as $TD_A(n) = (4\times n+\frac{29}{84})\times 16^n+\frac{32}{7}\times 2^n-\frac{8}{3}$, for $n\geq 3$. By Equations (3a), (3b), (4a), and (4b), Equation (2b) can be rewritten as $TD_M(n) = (2\times n-\frac{85}{42})\times 16^n+8\times 4^n-\frac{64}{7}\times 2^n+\frac{8}{3}$, for $n\geq 4$. Finally, Equation (1) becomes $TD(n) = (\frac{2}{9}\times n-\frac{113}{378})\times 16^{n+1}+(\frac{46}{9}\times n-\frac{70}{9})\times 4^{n+1}+\frac{192}{7}\times 2^{n+1}-\frac{40}{9}\times n-18\frac{26}{27}$ by Equation (2a) and (2b).

As mentioned earlier, the average distance of $PM[n]$ is equal to $\frac{9\times TD(n)}{16^{n+1}-5\times 4^{n+1}+4}$. Some values of average distance of $PM[n]$ are shown in Table 1 and Figure 8 is used to present the relations between diameter and average distance for pyramid networks. Notice that the differences between average distance and diameter of $PM[n]$ are nearly 2.690 for $n\geq 9$.

Table 1. Average Distance and Diameter of $PM[n]$

n	Diameter	Average distance
1	2	1.200
2	4	2.248
3	6	3.703
4	8	5.455
5	10	7.358
6	12	9.325
7	14	11.314
8	16	13.311
9	18	15.310
10	20	17.310
11	22	19.310
12	24	21.310
13	26	23.310

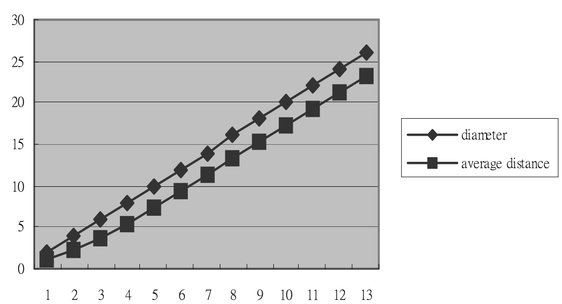


Figure 8. The comparison of diameter and average distance of $PM[n]$.

5. Conclusion

The pyramid network is one of the important architectures in parallel computing, network computing, and image processing. This investigation calculates the average distance of the pyramid network based on the known shortest path structure and distances. Because calculating the average distance of an irregular network is quite difficult, before this work, no articles investigated the average distance of a pyramid network.

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References:

[1] V. Cantoni, M. Ferretti, S. Levialdi, F. Maloberti, A pyramid project using integrated technology, in Integrated Technology for Parallel Image Processing, Academic Press, New York, 1985.

[2] F. Cao, D. Z. Du, D. F. Hsu, S. H. Teng, Fault tolerance properties of pyramid networks, IEEE Trans. Computers 48 (1) (1999), 88-93.

[3] Y. C. Chen, Pancycles and hamiltonian connectedness of the pyramid network with one node or one edge fault, Master Thesis, Department of Computer Science and Information Engineering, National Chi Nan University, Taiwan, 2003.

[4] L. Cinque, G. Bongiovanni, Parallel prefix computation on a pyramid computer, Pattern Recog. Lett. 16 (1) (1995) 19-22.

[5] R. Cipher, H. L. G. Sanz, SIMD architectures and algorithms for image processing and computer vision, IEEE Trans. Acoustic, Speech, and Signal Processing 37 (12) (1989) 2158-2174.

[6] D. E. Culler, J. P. Singh, A. Gupta, Parallel computer architecture: a hardware/software approach, Morgan Kaufmann, San Francisco, CA, 1999.

[7] D. R. Chen, C. C. Hsu, Fault-tolerant routing for pyramid networks using the least layer minimal routing method, Computer Systems: Science & Engineering 18 (1) (2003) 35-44.

[8] H.-J. Hsieh, D.-R. Duh, and J.-S. Shiau, A constant time shortest-path routing algorithm for pyramid networks, Proc. Int. Conf. on Parallel and Distributed Computing and Systems, MIT Cambridge, MA, USA, (2004) 71-75.

[9] J. F. Jenq, S. Sahni, Image shrinking and expanding on a pryamid, IEEE Trans. Parallel and Distributed Systems, 4 (11) (1993) 1291-1296.

[10] F. T. Leighton, Introduction to parallel algorithms and architectures: arrays, trees, hypercubes, Morgan Kaufmann, San Francisco, CA, 1991.

[11] R. Miller, Q. F. Stout, Pyramid computer algorithms for determining geometric properties of images, Proc. ACM Symposium on Computational Geometry (1985) 263-271.

[12] R. Miller, Q. F. Stout, Data movement techniques for the pyramid computer, SIAM J. on Computing, 16 (1) (1987) 38-60.

[13] M. J. Quinn, Designing efficient algorithms for parallel computers, McGraw-Hill, New York, 1987.

[14] H. Sarbazi-Azad, M. Ould-Khaoua, L. M. Mackenzie, Algorithmic construction of hamiltonians in pyramids, Inform. Proces. Lett. 80 (2) (2001) 75-79.

[15] Z. Shen, A routing algorithm for pyramid structures, Proc. ACM Symposium on Applied Computing, Las Vegas, Nevada, USA (2001) 484-488.

[16] Z. Shen, An alternative routing algorithm for the pyramid structures, Proc. ACM Symposium on Applied Computing, Melbourne, FL, USA (2003) 1009-1013.

[17] J. S. Shiau, Wide diameters of pyramid networks, Master Thesis, Department of Computer Science and Information Engineering, National Chi Nan University, Taiwan, 2004.

[18] Q. F. Stout, Drawing straight lines with a pyramid cellular automation, Inform. Process. Lett. 15 (5) (1982) 233-237.

[19] S. L. Tanimoto, A. Klinger, Structured computer vision: Machine Perception through Hierarchal computation Structure, Academic Press, New York, 1980.

[20] R. Y. Wu, D. R. Duh, Pancyclicity of pyramid networks, Proc. Int. Conf. on Parallel and Distributed Processing Techniques and Applications, Las Vegas, Nevada, USA (2004) 1523-1526.

[21] J. Xu, Topological structure and analysis of interconnection networks, Kluwer academic publishers, Dordrecht, Netherlands, 2002.